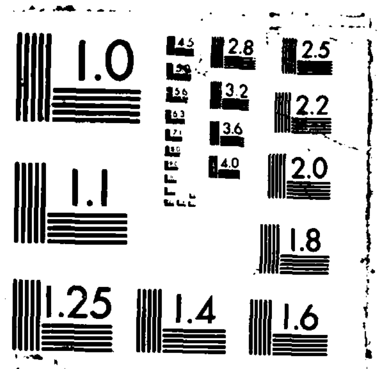
AD-A189 520 COHERENCE EFFECTS IN OPTICAL PHYSICS WITH SPECIAL REFERENCE TO SPECTROSCOPY(U) ROCHESTER UNIV NY DEPT OF PHYSICS AND ASTRONOMY E WOLF JAN 88 AFGL-TR-88-0006 1/1

UNCLASSIFIED F19628-85-K-0005 F/G 20/6 NL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AFGL-TR-88-0006

Coherence Effects in Optical Physics With Special Reference to Spectroscopy

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January 1988

Final Report
November 1984-January 1988

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Unclassified
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: Unclassified

1b. RESTRICTIVE MARKINGS:

2a. SECURITY CLASSIFICATION AUTHORITY:

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE:

3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; Distribution unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S):

5. MONITORING ORGANIZATION REPORT NUMBER(S): AFGL-TR-88-0006

6a. NAME OF PERFORMING ORGANIZATION: University of Rochester

6b. OFFICE SYMBOL (If applicable):

7a. NAME OF MONITORING ORGANIZATION: Air Force Geophysics Laboratory

6c. ADDRESS (City, State and ZIP Code): Department of Physics & Astronomy, Rochester, NY 14627

7b. ADDRESS (City, State and ZIP Code): Hanscom AFB, Massachusetts 01731

8a. NAME OF FUNDING/SPONSORING ORGANIZATION:

8b. OFFICE SYMBOL (If applicable):

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F19628-85-K-0005

8c. ADDRESS (City, State and ZIP Code):

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2310	2310G1	2310G1BG

11. TITLE (Include Security Classification): Coherence Effects in Optical Physics With Special Reference to Spectroscopy

12. PERSONAL AUTHOR(S): Emil Wolf

13a. TYPE OF REPORT: FINAL

13b. TIME COVERED: FROM Nov 1984 TO Jan 1988

14. DATE OF REPORT (Yr., Mo., Day): 1988 January

15. PAGE COUNT: 92

16. SUPPLEMENTARY NOTATION:

17. COSATI CODES

FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Coherence, Spectroscopy, Radiometry, Red Shift, Blue Shift

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Results are reported of investigations in the area of light propagation and spectroscopy from sources of arbitrary states of coherence. The research undertaken has completely clarified the foundations of radiometry in free space and also resolved the limitations of the radiometric model. It was demonstrated conclusively that spatial coherence properties of a source influence the nature of the spectrum of the emitted radiation. A number of results were derived relating to effects of a random medium on the state of polarization of light transmitted through it.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION: Unclassified

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22b. TELEPHONE NUMBER (Include Area Code): (617)377-3655

22c. OFFICE SYMBOL: OPI

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE

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NORTH-HOLLAND
PHYSICS
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REDSHIFTS AND BLUESHIFTS OF SPECTRAL LINES CAUSED BY SOURCE CORRELATIONS*

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Received 24 November 1986

We recently showed that the spectrum of light emitted by a source depends not only on the spectrum of the source distribution but also on the degree of spectral coherence of the source fluctuations. In this note we show that with a degree of spectral coherence of certain kind, specified by two parameters, the spectrum of the emitted light will be displaced relative to the source spectrum. The displacement will be either toward the lower or toward the higher frequencies, depending on the choice of the parameters.

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REDSHIFTS AND BLUESHIFTS OF SPECTRAL LINES CAUSED BY SOURCE CORRELATIONS☆

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We recently showed that the spectrum of light emitted by a source depends not only on the spectrum of the source distribution but also on the degree of spectral coherence of the source fluctuations. In this note we show that with a degree of spectral coherence of certain kind, specified by two parameters, the spectrum of the emitted light will be displaced relative to the source spectrum. The displacement will be either toward the lower or toward the higher frequencies, depending on the choice of the parameters.

1. Introduction

It has been known for some time that the spectrum of light generally changes on propagation, even in free space [1,2]. Such changes are basically due to correlation properties of the source. Recently we derived a condition for the normalized spectrum of light generated by a planar, secondary, quasi-homogeneous source to be the same throughout the far zone and in the source plane [3]. We referred to this condition, which is a requirement on the functional form of its degree of spectral coherence, as the scaling law and we noted that all quasi-homogeneous lambertian sources satisfy this law. We have also shown that when the scaling law is not satisfied the spectrum of the emitted light will, in general, no longer be invariant on propagation. These theoretical predictions have been recently verified by experiments [4].

In another recent paper [5] we considered radiation from three-dimensional, quasi-homogeneous sources and we showed that if the source spectrum consists of a line with a gaussian profile and if the degree of spectral coherence of the source is appropriately chosen, the spectrum of the emitted light will also consist of a line with gaussian profile, but this line will be redshifted with respect to the spectral line of the source distribution. The amount of the redshift depends on the spectral correlation length of the source. This result has important implications for astrophysics, some of which were briefly mentioned in ref. [5].

In the present note we again consider a source whose spectrum consists of a single line with a gaussian profile but we assume somewhat different correlation properties of the source. More specifically we choose a degree of spectral coherence of the source distribution which depends on two parameters rather than on a single parameter as we have done previously. The spectrum of the emitted light is again found to be a line with gaussian profile, but this line may be redshifted or blueshifted relative to the spectral line of the source distribution, depending on the choice of the parameters.

2. The spectrum of light produced by a three-dimensional quasi-homogeneous source

Let us consider a fluctuating source-distribution $Q(\boldsymbol{r}, t)$ occupying a finite domain of volume D in free space and let $V(\boldsymbol{r}, t)$ denote the field generated by the source. Here $\boldsymbol{r}$ denotes the position vector of a typical point and t the time. Both $V(\boldsymbol{r}, t)$ and $Q(\boldsymbol{r}, t)$ are taken to be analytic signals [6]. They are related by the inhomogeneous wave equation

☆ Research supported by the National Science Foundation under Grant PHY-8314626 and the Air Force Geophysics Laboratory under AFOSR Task 2310G1.

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$$\nabla^2 V(\mathbf{r}, t) - c^{-2}(\partial^2/\partial t^2)\, V(\mathbf{r}, t) = -4\pi\, Q(\mathbf{r}, t)\,. \tag{2.1}$$

We will assume that the statistical ensembles that characterize the source fluctuations are stationary. Let $W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)$ and $W_V(\mathbf{r}_1, \mathbf{r}_2, \omega)$ be the cross-spectral densities of the source distribution and of the field distribution respectively. They may be represented in the form [7]

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle U_Q^*(\mathbf{r}_1, \omega)\, U_Q(\mathbf{r}_2, \omega)\rangle\,, \tag{2.2a}$$

$$W_V(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle U_V^*(\mathbf{r}_1, \omega)\, U_V(\mathbf{r}_2, \omega)\rangle\,, \tag{2.2b}$$

where $\{U_Q(\mathbf{r}, \omega)\}$ and $\{U_V(\mathbf{r}, \omega)\}$ are ensembles of suitably chosen realizations, angular brackets denote averages taken over these ensembles and the asterisk denotes the complex conjugate. As consequence of the wave equation (2.1) the two cross-spectral densities may be shown to be related by the equation [‡1] [ref. [7], eq. (3.10); ref. [8a], eq. (2.11)]

$$(\nabla_1^2 + k^2)\,(\nabla_2^2 + k^2)\, W_V(\mathbf{r}_1, \mathbf{r}_2, \omega) = (4\pi)^2\, W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega)\,, \tag{2.3}$$

where ∇_1^2 and ∇_2^2 are the laplacian operators acting with respect to the coordinates of the points $\mathbf{r}_1$ and $\mathbf{r}_2$ *respectively and*

$$k = \omega/c \tag{2.4}$$

is the wave number associated with the frequency ω, c being the speed of light in vacuo.

Using eq. (2.3) one can show that the radiant intensity $J_\omega(\mathbf{u})$ generated by the source, i.e. the rate at which energy is radiated at frequency ω per unit solid angle around a direction specified by a unit vector $\mathbf{u}$ is given by [ref. [8a], eq. (3.9)]

$$J_\omega(\mathbf{u}) = (2\pi)^6\, \tilde{W}_Q(-k\mathbf{u}, k\mathbf{u}, \omega)\,, \tag{2.5}$$

where

$$\tilde{W}_Q(\mathbf{K}_1, \mathbf{K}_2, \omega) = (2\pi)^{-6} \int_D\int_D W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) \times \exp[-\mathrm{i}(\mathbf{K}_1\cdot\mathbf{r}_1 + \mathbf{K}_2\cdot\mathbf{r}_2)]\, \mathrm{d}^3 r_1\, \mathrm{d}^3 r_2 \tag{2.6}$$

is the six-dimensional Fourier transform of W_Q.

We will restrict our attention to quasi-homogeneous sources. For such sources one has, to a good approximation,

$$W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) = S_Q[(\mathbf{r}_1 + \mathbf{r}_2)/2, \omega]\, \mu_Q(\mathbf{r}_2 - \mathbf{r}_1, \omega)\,, \tag{2.7}$$

where

$$S_Q(\mathbf{r}, \omega) \equiv W_Q(\mathbf{r}, \mathbf{r}, \omega) = \langle U_Q^*(\mathbf{r}, \omega)\, U_Q(\mathbf{r}, \omega)\rangle \tag{2.8}$$

is the source spectrum and

$$\mu_Q(\mathbf{r}_2 - \mathbf{r}_1, \omega) \equiv W_Q(\mathbf{r}_1, \mathbf{r}_2, \omega) \times [S_Q(\mathbf{r}_1, \omega)]^{-1/2}\, [S_Q(\mathbf{r}_2, \omega)]^{-1/2} \tag{2.9}$$

is the degree of spectral coherence of the source distribution. Moreover, for each effective frequency ω contained in the source spectrum, $S_Q(\mathbf{r}, \omega)$ varies much more slowly with $\mathbf{r}$ than $\mu_Q(\mathbf{r}', \omega)$ varies with $\mathbf{r}'$. With sources of this class eq. (2.5) takes the form [ref. [8b], eq. (3.11)]

$$J_\omega(\mathbf{u}) = (2\pi)^6 \tilde{S}_Q(0, \omega)\tilde{\mu}_Q(k\mathbf{u}, \omega)\,, \tag{2.10}$$

where the tilde now denotes three-dimensional Fourier transforms.

Let us next assume that the source spectrum is the same at each source point. We will then write $S_Q(\omega)$ in place of $S_Q(\mathbf{r},\omega)$. In this case $\tilde{S}_Q(0, \omega) = DS_Q(\omega)/(2\pi)^3$ and the formula (2.10) becomes (D again denoting the source volume)

$$J_\omega(\mathbf{u}) = (2\pi)^3 DS_Q(\omega)\tilde{\mu}_Q(k\mathbf{u}, \omega)\,. \tag{2.11}$$

Now the radiant intensity $J_\omega(\mathbf{u})$ is trivially related to the spectrum $S_V^{(\infty)}(R\mathbf{u}, \omega) \equiv W_V^{(\infty)}(R\mathbf{u}, R\mathbf{u}, \omega)$ of the far field by the formula [9] $S_V^{(\infty)}(R\mathbf{u}, R\mathbf{u}, \omega) \sim J_\omega(\mathbf{u})/R^2$ as $kR \to \infty$, with the unit vector $\mathbf{u}$ fixed. Hence we obtain at once from eq. (2.11) the following expression for the spectrum of the emitted light in the far zone:

$$S_V^{(\infty)}(\mathbf{u}, \omega) = (2\pi)^3 (D/R^2)\, S_Q(\omega)\, \tilde{\mu}_Q(k\mathbf{u}, \omega)\,. \tag{2.12}$$

This formula shows that the spectrum $S_V^{(\infty)}(\mathbf{u}, \omega)$ of the emitted light in the far zone depends, in general, not only on the source spectrum $S_Q(\omega)$ but also on

[‡1] The definition of the cross-spectral densities employed in refs. [7] and [8] differ by complex conjugation. Throughout this note we employ those of ref. [7]; hence some of the formulas we now use [e.g. eq. (2.5) below] differ trivially from the corresponding formulas of refs. [8].

the degree of spectral coherence of the source distribution. It seems worthwhile to note that the dimensions of $S_1^{(\infty)}$ and of S_Q are different. Since $\tilde{\mu}_Q$ is the three-dimensional Fourier transform of μ_Q, $[\tilde{\mu}_Q] = L^3$ (brackets denoting dimensions and L denotes length). Hence eq. (2.12) implies that $[S_1^{(\infty)}] = [S_Q]L^4$, in agreement with eq. (2.3).

3. A class of source correlations that generate lineshifts

In ref. [5] we considered quasi-homogeneous sources whose spectrum was a line of gaussian profile,

$$S_Q(\omega) = A\exp[-(\omega-\omega_0)^2/2\delta_0^2]\,, \quad (\delta_0/\omega_0 \ll 1)\,, \tag{3.1}$$

and whose degree of spectral coherence was also gaussian viz.,

$$\mu_Q(\boldsymbol{r}', \omega) = \exp[-r'^2/2\sigma^2(\omega)]\,, \tag{3.2}$$

where $r' = |\boldsymbol{r}'|$. The three-dimensional Fourier transform of μ_Q is then given by

$$\tilde{\mu}_Q(\boldsymbol{K}, \omega) = [\sigma(\omega)/\sqrt{2\pi}]^3 \exp[-\tfrac{1}{2}K^2\sigma^2(\omega)]\,, \tag{3.3}$$

($K = |\boldsymbol{K}|$). In particular we showed that if $\sigma(\omega)$ is constant (ζ say) such a source will emit light whose spectrum in the far zone is redshifted with respect to the source spectrum, the amount of the shift depending on the effective source correlation length ζ.

The degrees of spectral coherence of the form (3.2), with $\sigma(\omega) = \zeta$ (constant) form a one-parameter family. In this note we will consider quasi-homogeneous sources whose degrees of coherence are of a somewhat more general form. Specifically we assume that for these sources

$$\tilde{\mu}_Q(\boldsymbol{K}) = B\exp[-\tfrac{1}{2}(K-K_1)^2\zeta^2]\,, \tag{3.4}$$

where B, K_1 and ζ are positive constants. We have written $\tilde{\mu}_Q(\boldsymbol{K})$ rather than $\tilde{\mu}_Q(\boldsymbol{K}, \omega)$ on the left-hand side of eq. (3.4), because $\tilde{\mu}_Q$ is now independent of ω. Only two of the three constants in the expression (3.4) are independent, because the Fourier transform $\mu_Q(\boldsymbol{r}')$ of $\tilde{\mu}_Q(\boldsymbol{K})$ satisfies the requirement that $\mu_Q(0) = 1$, which is a necessary condition for $\mu_Q(\boldsymbol{r}')$ to be a correlation coefficient.

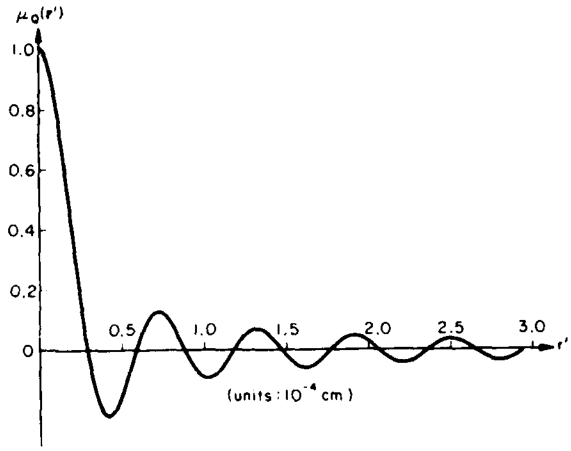

Fig. 1. The behaviour of the correlation coefficient $\mu_Q(r') = [(\sin K_1 r')/K_1 r']\exp(-r'^2/2\zeta^2)$, with $K_1 = 1.07\times10^5$ cm^{-1}, $\zeta = 1.5$ cm [associated with curve (d) in fig. 2].

It can be shown by a long but straightforward calculation *(which we omit because of limitation of space)* that if

$$K_1\zeta \gg 1 \tag{3.5}$$

the degree of spectral coherence, whose Fourier transform is given by eq. (3.4), is

$$\mu_Q(\boldsymbol{r}') = [(\sin K_1 r')/K_1 r']\ \exp(-r'^2/2\zeta^2) \tag{3.6}$$

and that the constant B in eq. (3.4) is given in terms of the two other parameters by the formula

$$B = \zeta/2(2\pi)^{1/2}K_1^2\,. \tag{3.7}$$

From now on we will only consider situations for which the constraint (3.5) holds. Eq. (3.6) then shows that the degree of spectral coherence has the form of the sinc function $(\sin K_1 r')/K_1 r'$, modulated by the gaussian function $\exp(-r'^2/2\zeta^2)$. The behaviour of such a two-parameter correlation coefficient is shown in fig. 1.

It follows on substituting from eqs. (3.1) and (3.4) into eq. (2.12) that the spectrum of the light in the far zone, generated by such a source, is given by

$$S_1^{(\infty)}(\omega) = (2\pi)^3(D/R^2)AB\exp[-(\omega-\omega_0)^2/2\delta_0^2] \times \exp[-(\omega-\omega_1)^2/2\delta_1^2]\,, \tag{3.8}$$

where $\omega = kc$ as before and

$$\omega_1 = K_1 c, \quad \delta_1 = c/\zeta . \tag{3.9}$$

We have written $S_1^{(\infty)}(\omega)$ rather than $S_1^{(\infty)}(\mathbf{u}, \omega)$ on the left-hand side of eq. (3.8) since, because of the assumed isotropy of the source, $S_1^{(\infty)}$ is now independent of $\mathbf{u}$. In terms of the parameters ω_1 and δ_1 the factor B, given by eq. (3.7), becomes

$$B = c^3/2(2\pi)^{3/2}\omega_1^2\delta_1 . \tag{3.10}$$

Let us now consider the expression (3.8) more closely. For this purpose it is convenient to set

$$\alpha_0 = 1/2\delta_0^2, \quad \alpha_1 = 1/2\delta_1^2 . \tag{3.11}$$

One then finds after a straightforward calculation that eq. (3.8) may be expressed in the form

$$S_1^{(\infty)}(\omega) = AC \exp[-(\omega - \omega_{01})^2/2\delta_{01}^2] , \tag{3.12}$$

where

$$\omega_{01} = (\alpha_0\omega_0 + \alpha_1\omega_1)/(\alpha_0 + \alpha_1) , \tag{3.13}$$

$$1/\delta_{01}^2 = 2(\alpha_0 + \alpha_1) = (1/\delta_0^2) + (1/\delta_1^2) , \tag{3.14}$$

and

$$C = (2\pi)^3(DB/R^2) \times \exp\{-[\alpha_0\alpha_1/(\alpha_0 + \alpha_1)](\omega_1 - \omega_0)^2\} . \tag{3.15}$$

The formula (3.12) shows that the spectrum of the emitted light in the far zone is also a line with gaussian profile, but it is not centered on the frequency ω_0 of the source spectrum [cf. eq. (3.1)] but rather on the frequency ω_{01}, given by eq. (3.13). Since according to eqs. (3.11) α_0 and α_1 are positive constants one can readily deduce from the expression (3.13) that

$$\omega_{01} < \omega_0 \quad \text{when } \omega_1 < \omega_0 ,$$

and that

$$\omega_{01} > \omega_0 \quad \text{when } \omega_1 > \omega_0 .$$

Since according to eq. (3.9) $\omega_1 = K_1 c$, this result implies that *if the parameter K_1 of the degree of spectral coherence (3.6) is smaller than the wavenumber $k_0 = \omega_0/c$ associated with the source spectrum $S_Q(\omega)$, the spectrum $S_1^{(\infty)}(\omega)$ of the emitted light is redshifted with respect to $S_Q(\omega)$; and that if K_1 is greater than k_0 it is blueshifted with respect to it.* We also see from eq. (3.14) that $1/\delta_{01}^2 > 1/\delta_0^2$ i.e. that $\delta_{01} < \delta_0$. Hence in either case the spectral line of the emitted light is narrower than the spectral line of the source distribution.

4. Examples

To illustrate the preceding analysis we consider a few examples. For simplicity we will choose

$$\delta_1 = \delta_0 . \tag{4.1}$$

Then, according to eq. (3.11), $\alpha_1 = \alpha_0$ and the expression (3.12) becomes

$$S_1^{(\infty)}(\omega) = A\bar{C} \exp[-(\omega - \bar{\omega})^2/\delta_0^2] , \tag{4.2}$$

where

$$\bar{\omega} = \tfrac{1}{2}(\omega_0 + \omega_1) , \tag{4.3}$$

$$\bar{C} = (2\pi)^3(DB/R) \exp[-(\omega_1 - \omega_0)^2/4\delta_0^2]. \tag{4.4}$$

We see that the spectral line of the emitted light is now centered on the average value $\bar{\omega}$ of the frequencies ω_0 and ω_1.

Let us consider the normalized spectrum

$$s_1^{(\infty)}(\omega) = S_1^{(\infty)}(\omega) \Big/ \int_0^\infty S_1^{(\infty)}(\omega)\, d\omega \tag{4.5}$$

of the emitted light. On substituting from eq. (4.2) into eq. (4.5) and on using eq. (4.3) we obtain the following expression for $s_1^{(\infty)}(\omega)$:

$$s_1^{(\infty)}(\omega) = (1/\delta_0\sqrt{\pi}) \times \exp\{-[\omega - \tfrac{1}{2}(\omega_0 + \omega_1)]^2/\delta_0^2\} . \tag{4.6}$$

In fig. 2 curves are plotted showing the normalized source spectrum

$$s_Q(\omega) = (1/\delta_0\sqrt{2\pi}) \exp[-(\omega - \omega_0)^2/2\delta_0^2] , \tag{4.7}$$

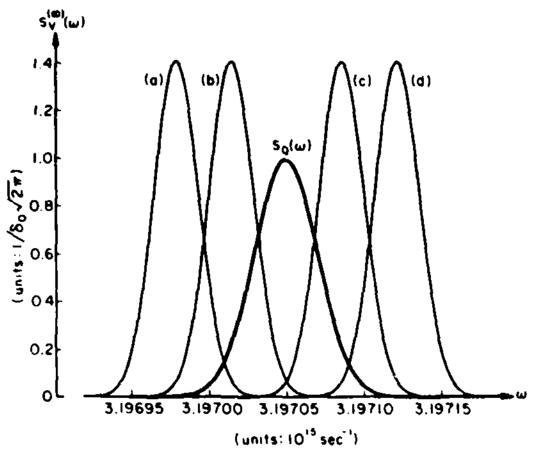

Fig. 2. Redshifts and blueshifts of spectral lines caused by source correlations. The normalized spectrum $s_Q(\omega)$ of the source distribution is a line of gaussian profile [given by eq. (4.7)], with $\omega_0 = 3.197049 \times 10^{15}$ s^{-1} (sodium line of wavelength $\lambda_0 = 5895.924$ Å) and rms width $\delta = 2\times 10^{10}$ s^{-1}. Curves (a) – (d) show the normalized spectra of the emitted light [lines with gaussian profiles given by eq. (4.6)], generated by the source distribution, each with $\delta_1 = \delta_0$ ($\zeta = c/\delta_1 = 1.5$ cm) and with $\omega_1 = \omega_0 - 1.4\times 10^{11}$ s^{-1} (a), $\omega_1 = \omega_0 - 0.7\times 10^{11}$ s^{-1} (b), $\omega_1 = \omega_0 + 0.7\times 10^{11}$ s^{-1} (c) and $\omega_1 = \omega_0 + 1.4\times 10^{11}$ s^{-1} (d).

taken to be one of the sodium lines, as well as a number of emitted lines for different values of the parameter $\omega_1 = K_1 c$, of the degree of spectral coherence of the source; the other parameter, ζ, is kept fixed and chosen so that $\delta_1 = c/\zeta$ is equal to δ_0. It is seen that with increasing values of the difference $|\omega_0 - \omega_1|$ the shift of the emitted spectral line also increases. This, of course, is to be expected since when $\delta_1 = \delta_0$, the shift is given by $|\bar{\omega} - \omega_0| = \frac{1}{2}|\omega_0 - \omega_1|$.

Acknowledgements

I am obliged to Prof. G.S. Agarwal and to Mr. K. Kim for helpful discussions and to Mr. A. Gamliel for carrying out the computations relating to figs. 1 and 2.

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RADIATION EFFICIENCY OF PLANAR GAUSSIAN SCHELL-MODEL SOURCES*

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Received 2 September 1986

A general expression is derived for the ratio of the radiated power and the source-integrated intensity for any planar gaussian Schell-model source. The behavior of this quantity, known as the radiation efficiency of the source, is displayed graphically as a function of the rms width of the intensity profile and the spatial coherence length of the light distribution across the source. Some limiting cases are discussed and it is shown that a gaussian-correlated quasi-homogeneous source may have higher radiation efficiency than a fully coherent Schell-model source with a gaussian intensity profile (e.g. a single mode laser).

1. Introduction

In the last few years there has been considerable interest in radiation produced by partially coherent sources. In particular, the radiation efficiency of sources of different states of coherence have been investigated. The radiation efficiency of a source is defined as the ratio of the total outgoing flux to the source-integrated intensity. The radiation efficiency of planar quasi-homogeneous sources was studied by Carter and Wolf [1,2]. More recently the radiation efficiency of three-dimensional gaussian Schell-model sources was calculated, and was compared with the radiation efficiency of a corresponding coherent source [3].

In the present paper we extend the analysis of Carter and Wolf to the important class of planar gaussian Schell-model sources. We derive an explicit expression for the radiation efficiency of sources of this class and present diagrams which show its dependence on the rms widths of the intensity profile and its degree of coherence. We also examine homogeneous sources, completely coherent sources, and quasi-homogeneous sources as limiting cases of gaussian Schell-model sources. Finally we derive conditions under which a planar gaussian-correlated Schell-model source is more efficient than a completely coherent source with a gaussian intensity profile (e.g. a single mode laser).

* Research supported by the Air Force Geophysics Laboratory under AFOSR Task 2310G1 and by the Army Research Office.

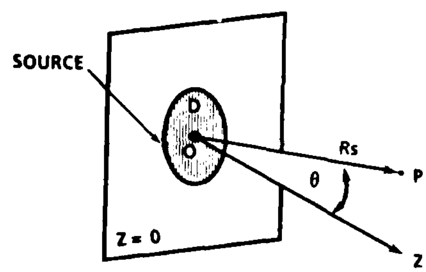

Fig. 1. Illustration of the notation. P represents a field point in the far zone.

2. The radiation efficiency of a planar gaussian Schell-model source

Consider a planar secondary Schell-model source occupying a domain D in the plane $z=0$ and radiating into the half-space $z>0$ (see fig. 1). Such a source is characterized by a cross spectral density function of the form [4]

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = [I(\boldsymbol{r}_1, \omega)\, I(\boldsymbol{r}_2, \omega)]^{1/2} g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \omega) , \quad (1)$$

where $I(\boldsymbol{r}, \omega)$ is the intensity profile and $g(\boldsymbol{r}_1 - \boldsymbol{r}_2,$

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ω) is the complex degree of spatial coherence, both taken at frequency ω. The symbols $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are position vectors of typical points in the source region D.

It is known that the radiant intensity produced by a secondary planar source in the direction specified by a unit vector $\boldsymbol{s}$ is given by [5]

$$J_\omega(\boldsymbol{s}) = (2\pi k)^2 \cos^2\theta \; \tilde{W}(k\boldsymbol{s}_\perp, -k\boldsymbol{s}_\perp, \omega) . \tag{2}$$

Here $k=\omega/c$ ($c=$ speed of light in vacuum) is the wave number associated with frequency ω, θ is the angle between the $\boldsymbol{s}$ direction and the normal to the source plane, and

$$\tilde{W}(\boldsymbol{f}_1, \boldsymbol{f}_2, \omega) = (2\pi)^{-4} \iint_{-\infty}^{\infty} W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$$

$$\times \exp[-\mathrm{i}(\boldsymbol{f}_1\cdot\boldsymbol{r}_1 + \boldsymbol{f}_2\cdot\boldsymbol{r}_2)] \, \mathrm{d}^2 r_1 \, \mathrm{d}^2 r_2 \tag{3}$$

is the four-dimensional spatial Fourier transform of the cross-spectral density of the light distribution in the source plane, with $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ representing two-dimensional spatial-frequency vectors.

The total flux emitted by the source into the half-space $z>0$ is given by the expression

$$\Phi_\omega = \int_{(2\pi)} J_\omega(\boldsymbol{s}) \, \mathrm{d}\Omega , \tag{4}$$

where the symbol (2π) under the integral sign indicates that the integration is taken over the solid angle subtended by a hemisphere in the half-space $z>0$, centered at the origin.

We define the radiation efficiency of a source [cf. ref. 2. eq. (3.11)] by the formula

$$\epsilon_\omega = \Phi_\omega \Big/ \int I(r, \omega) \mathrm{d}^2 r . \tag{5}$$

The integration in the denominator of eq. (5) is taken over the source domain D. We show in the Appendix that $\epsilon_\omega \leq 1$ for any planar source.

We will now consider planar Schell-model sources for which both the intensity distribution and the degree of spatial coherence are gaussian, i.e. they have the form

$$I(\boldsymbol{r}, \omega) = I_0 \exp(-r^2/2\sigma_I^2) , \tag{6a}$$

and

$$g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \omega) = \exp[-(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2/2\sigma_g^2] . \tag{6b}$$

Here I_0, σ_g and σ_g are positive quantities depending only on the frequency ω (dependence not displayed). Such sources are known as gaussian Schell-model sources.

On substituting from eqs. (6) into eq. (1) and taking the four-dimensional spatial Fourier transform one can show after a lengthy calculation that [cf. refs. 6 and 7]

$$\tilde{W}(k\boldsymbol{s}_\perp, -k\boldsymbol{s}_\perp, \omega)$$

$$= \frac{I_0}{4(2\pi)^2 \alpha^2 (\alpha^2 + 2\beta^2)} \exp\left(-\frac{k^2 \sin^2\theta}{2(\alpha^2 + 2\beta^2)}\right) , \tag{7}$$

where

$$\alpha^2 = 1/(4\sigma_I^2) , \quad \beta^2 = 1/(2\sigma_g^2) . \tag{8}$$

Next if we substitute from eq. (7) into eq. (2) we obtain the following expression for the radiant intensity generated by a source of the type we are considering:

$$J_\omega(\boldsymbol{s}) = \frac{k^2 I_0}{4\alpha^2(\alpha^2 + 2\beta^2)} \cos^2\theta$$

$$\times \exp\left(-\frac{k^2 \sin^2\theta}{2(\alpha^2 + 2\beta^2)}\right) . \tag{9}$$

It follows on substituting this expression into eq. (4), that the total flux at frequency ω radiated by a planar gaussian Schell-model source into the half-space $z>0$ is given by

$$\Phi_\omega = \frac{k^2 I_0}{4\alpha^2(\alpha^2 + 2\beta^2)} \int_{(2\pi)} \cos^2\theta$$

$$\times \exp\left(-\frac{k^2}{2(\alpha^2 + 2\beta^2)} (1 - \cos^2\theta)\right) \mathrm{d}\Omega . \tag{10}$$

After some algebraic manipulation this expression can be reduced to

$$\Phi_\omega = 2\pi\sigma_I^2 I_0 \left(1 - \frac{\exp(-\xi^2)}{\xi} \int_0^\xi \exp(t^2) \, \mathrm{d}t\right) , \tag{11}$$

where

$$\xi^2 = [1/2(k\sigma_I)^2 + 2/(k\sigma_g)^2]^{-1} . \tag{12}$$

The denominator in eq. (5) with $I(\boldsymbol{r}, \omega)$ given by eq. (6a) can also be readily evaluated and we find that

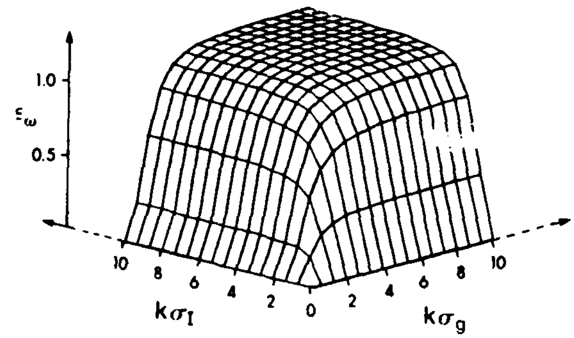

Fig. 2. Radiation efficiency ϵ_ω as a function of $k\sigma_g$ and $k\sigma_I$.

$$\int I(\mathbf{r},\omega)\,\mathrm{d}^2r = 2\pi I_0\sigma_I^2\,. \tag{13}$$

On substituting from eqs. (11) and (13) into eq. (5) we finally obtain the following expression for the radiation efficiency of a planar gaussian Schell-model source:

$$\epsilon_\omega = 1 - D(\xi)/\xi\,, \tag{14}$$

where

$$D(\xi) = \exp(-\xi^2)\int_0^\xi \exp(t^2)\mathrm{d}t \tag{15}$$

is the Dawson integral [8].

Fig. 2 shows a three-dimensional plot of the radiation efficiency ϵ_ω as a function of $k\sigma_g$ and $k\sigma_I$ calculated from eqs. (14) and (12). Fig. 3(a) shows the behavior of the radiation efficiency as a function of $k\sigma_g$ and fig. 3(b) shows its behavior as a function of $k\sigma_I$ for some selected values of the other parameter.

3. Physical interpretation

As can be seen from eq. (14) the radiation efficiency ϵ_ω depends on the rms widths of the intensity profile and of the degree of spatial coherence only through the parameter ξ defined by eq. (12). A consequence of this fact is an equivalence theorem for the radiation efficiency: there exist an infinite number of planar gaussian Schell-model sources of different rms intensity width σ_I and different spectral coherence lengths σ_g which have the same radiation

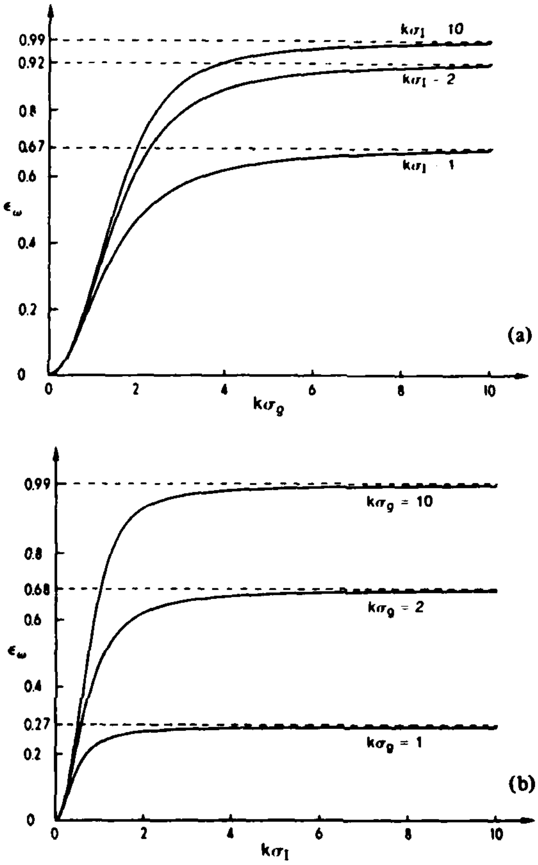

Fig. 3. Radiation efficiency ϵ_ω as a function of $k\sigma_g$ for selected values of $k\sigma_I$ (a) and as a function of $k\sigma_I$ for selected values of $k\sigma_g$ (b).

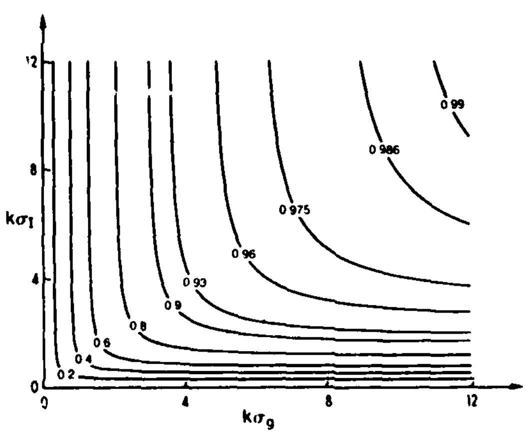

Fig. 4. Contours of equal radiation efficiency as a function of $k\sigma_g$ and $k\sigma_I$.

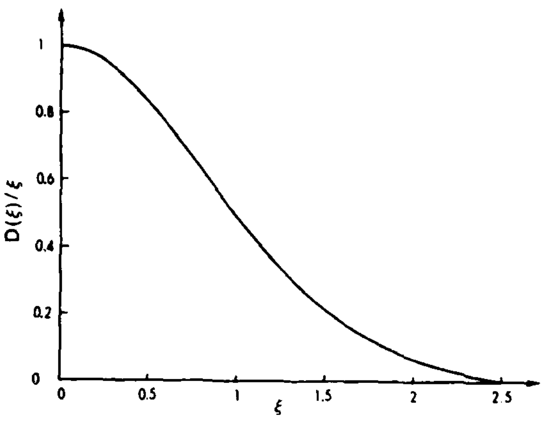

Fig. 5. Graphical representation of $D(\xi)/\xi$, where $D(\xi)$ is the Dawson integral (15).

efficiency ϵ_ω. For a given value of ϵ_ω a class of equivalent sources is represented by a single curve in fig. 4.

We shall now consider a number of limiting cases that are of special interest.

3.1. The coherent limit ($k\sigma_g \to \infty$)

When the source of the class that we are considering is completely coherent, $k\sigma_g \to \infty$ and eq. (12) implies that $\xi \to \sqrt{2}k\sigma_I$. The expression (14) for the radiation efficiency then becomes

$$\epsilon_\omega = 1 - D(\sqrt{2}k\sigma_I)/\sqrt{2}k\sigma_I \,. \quad (16)$$

Since the second term on the right of eq. (16) approaches zero as $k\sigma_I \to \infty$ (see fig. 5) we see that the radiation efficiency of a coherent source then approaches the value unity. The formula (16) applies to certain types of lasers operating in their lowest-order mode.

3.2. A homogeneous Schell-model source ($k\sigma_I \to \infty$)

Another interesting limiting case is obtained by letting $k\sigma_I \to \infty$ (with k being fixed), and $k\sigma_g$ having an arbitrary but fixed value. Eq. (6a) reduces to $I(r, \omega) = I_0$, and if we also make use of eq. (6b) the expression (1) for the spectral density of the source becomes

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = I_0 \exp[-(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2/2\sigma_g^2] \,. \quad (17)$$

Since $W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$ now depends on $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ only through the difference $\boldsymbol{r}_1 - \boldsymbol{r}_2$, the source is *homogeneous*. It follows from eq. (12) that as $k\sigma_I \to \infty$, $\xi \to k\sigma_g/\sqrt{2}$ and hence the expression (14) for the radiation efficiency now becomes

$$\epsilon_\omega = 1 - \frac{D(k\sigma_g/\sqrt{2})}{k\sigma_g/\sqrt{2}} \,. \quad (18)$$

Furthermore, on inspecting eq. (18) and using the fact that the function $D(\xi)/\xi$ monotonically decreases from unity to zero as ξ increases from zero to infinity (see fig. 5) we see that for radiation from a homogeneous gaussian Schell-model source, ϵ_ω increases monotonically with increasing $k\sigma_g$ and asymptotically approaches the value unity as $k\sigma_g \to \infty$. This limiting case corresponds to the situation where the field generated by the source coincides with a wavefront of a plane-wave field that propagates in the positive z-direction.

3.3. The quasi-homogeneous limit ($\sigma_I \gg \sigma_g$)

When $k\sigma_I \gg k\sigma_g$ a gaussian Schell-model source reduces to a gaussian correlated quasi-homogeneous source with a gaussian intensity profile. The radiation efficiency of such sources was shown by Carter and Wolf [1] to be given by

$$\epsilon_\omega = 1 - \frac{D(k\sigma_g/\sqrt{2})}{k\sigma_g/\sqrt{2}} \,. \quad (19)$$

It is clear that our expression (14), together with eq. (12), indeed reduce to eq. (19) in this limiting case.

We may also consider the limiting case of a completely coherent quasi-homogeneous source by letting $k\sigma_g \to \infty$, $k\sigma_I \to \infty$ with $k\sigma_I/k\sigma_g = \text{const.} \gg 1$. Since $D(\xi)/\xi \to 0$ as $\xi \to \infty$ it follows from eq. (19) that in this limit

$$\epsilon_\omega \to 1 \,. \quad (20)$$

Hence the radiation efficiency of a coherent quasi-homogeneous source is unity.

Finally we deduce from eqs. (16) and (19), if we recall once again that $D(\xi)/\xi$ decreases monotonically with increasing ξ, that when

$$(\sigma_g)_{\mathrm{q.h.}} > 2(\sigma_I)_{\mathrm{coh.}} \quad (21)$$

the radiation efficiency (at frequency ω) of a gaussian correlated quasi-homogeneous source will be greater than that of a completely coherent Schell-model with gaussian intensity profile. As we have mentioned above, certain types of lasers operating in their lowest order mode correspond to a coherent Schell-model source with a gaussian intensity profile. It is therefore clear from eq. (21) that a gaussian-correlated quasi-homogeneous source may have higher radiation efficiency than a coherent laser source emitting radiation of a gaussian intensity profile.

Appendix

Proof that $\epsilon_\omega \leq 1$ for planar Schell-model sources

It was shown in ref. [1] that for quasi-homogeneous sources $\epsilon_g \leq 1$. We will now show that this inequality holds, in fact, for all planar sources.

We start by showing that $\tilde{W}(\boldsymbol{f}, -\boldsymbol{f}, \omega) \geqslant 0$ for all real two-dimensional vectors $\boldsymbol{f}(0 \leqslant |\boldsymbol{f}| < \infty)$. The cross spectral density $W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$ is known to be non-negative definite [9] i.e.

$$\iint W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) f(\boldsymbol{r}_1) f^*(\boldsymbol{r}_2)\, \mathrm{d}^2 r_1\, \mathrm{d}^2 r_2 \geq 0\,, \quad (\mathrm{A1})$$

with any arbitrary function $f(\boldsymbol{r})$ for which the double integral converges. Let us choose $f(\boldsymbol{r}) = \exp(-\mathrm{i}\boldsymbol{f}\cdot\boldsymbol{r})$. The inequality (A1) then gives

$$\iint W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$$
$$\times \exp[-\mathrm{i}\boldsymbol{f}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)]\, \mathrm{d}^2 r_1\, \mathrm{d}^2 r_2 \geq 0\,, \quad (\mathrm{A2})$$

which implies at once [cf. eq. (3)] that

$$\tilde{W}(k\boldsymbol{s}_\perp, -k\boldsymbol{s}_\perp, \omega) \geq 0 \quad (0 \leq k|\boldsymbol{s}_\perp| \leq \infty)\,. \quad (\mathrm{A3})$$

If we substitute eq. (4) into eq. (5) and use the expression (2) for $J_\omega(\boldsymbol{s})$ we obtain the following formula for the radiation efficiency:

$$\epsilon_\omega = (2\pi k)^2 \int_{(2\pi)} \cos^2\theta\, \tilde{W}(k\boldsymbol{s}_\perp, -k\boldsymbol{s}_\perp, \omega)\mathrm{d}\Omega$$
$$\times \left(\int I(\boldsymbol{r}, \omega) \mathrm{d}^2 r \right)^{-1}. \quad (\mathrm{A4})$$

If we next make use of the relation $\cos^2\theta\, \mathrm{d}\Omega = (1 - s_x^2 - s_y^2)^{1/2} \mathrm{d}s_x \mathrm{d}s_y$ in eq. (A4) and recall that the intensity $I(\boldsymbol{r}, \omega) = W(\boldsymbol{r}, \boldsymbol{r}, \omega)$, we find that

$$\epsilon_\omega \leq (2\pi k)^2 \iint_{s_x^2 + s_y^2 \leq 1} W(k\boldsymbol{s}_\perp, -k\boldsymbol{s}_\perp, \omega)\, \mathrm{d}s_x\, \mathrm{d}s_y$$
$$\times \left(\int W(\boldsymbol{r}, \boldsymbol{r}, \omega)\, \mathrm{d}^2 r \right)^{-1}. \quad (\mathrm{A5})$$

In view of the inequality (A3) we may replace the integration over the unit circle in the numerator of eq. (A5) by integration over the whole s_x, s_y-plane. After doing so we substitute for $W(k\boldsymbol{s}_\perp, -k\boldsymbol{s}_\perp, \omega)$ from eq. (3) and interchange the orders of integrations. We then obtain the inequality

$$\epsilon_\omega \leq \iint_{-\infty}^{\infty} W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)\, \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\, \mathrm{d}^2 r_1\, \mathrm{d}^2 r_2$$
$$\times \left(\int W(\boldsymbol{r}, \boldsymbol{r}, \omega)\, \mathrm{d}^2 r \right)^{-1}, \quad (\mathrm{A6})$$

where δ is the Dirac delta function. On carrying out the trivial integration with respect to $\boldsymbol{r}_2$, we finally obtain the inequality

$$\epsilon_\omega \leq 1\,, \quad (\mathrm{A7})$$

valid for all planar sources.

Acknowledgement

The author wishes to express his thanks to Professor E. Wolf for helpful discussions and guidance concerning the topic of this paper and to Professor N. George for providing excellent facilities for this research.

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JOURNAL OF MODERN OPTICS, 1987, VOL. 34, NO. 4, 569–575

Mueller matrices and depolarization criteria

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(*Received 16 September 1986*)

Abstract. The question of whether a given Mueller matrix represents a deterministic or a non-deterministic system is analysed by means of a matrix condition. The possibility of replacing this matrix condition by a scalar condition is examined. It is shown that this is permissible only for those cases where a Hermitian matrix constructed from the Mueller matrix is positive semidefinite.

1. Introduction

Several methods have been used in the description of the polarization state of a wavefield. While the Jones method [1] and the Poincaré sphere method [2] are useful for the description of fully polarized states, the coherency matrix method [3] and the Mueller–Stokes method [4, 5] can handle both partially and fully polarized light. It should be noted that all these methods assume the radiation field under consideration to be an ensemble of plane waves all having the same wave-vector. It is only relatively recently that a systematic procedure for handling polarization in a beam field has been developed [6]. In the following we assume, however, the radiation field is of the former type.

The coherency matrix and the Stokes vector are equivalent, and carry exactly the same amount of information. However, when the passage of the beam through an optical system is encountered, the situation becomes quite different: the usual transformation law of the coherency matrix via the Jones matrix of the optical element corresponds to deterministic (non-depolarizing) systems; while the transformation of the Stokes vector through the Mueller matrix corresponds to more general systems including non-deterministic (depolarizing) systems. In the deterministic case the Mueller matrix can be derived from the Jones matrix of the system. A non-deterministic system, on the other hand, has a well-defined Mueller matrix; but there does not exist a Jones matrix from which it can be derived. This is to be expected, for the Jones matrices form a seven-parameter family (the absolute phase of the Jones matrix should be suppressed in any comparison with the Mueller matrix since it does not affect the transformation of the coherency matrix, this transformation being quadratic in the Jones matrix), whereas the Mueller matrices form a sixteen-parameter family.

In view of this situation the following question is of much practical interest. How can one determine whether an experimentally measured Mueller matrix corresponds to a deterministic or a non-deterministic system? This question was first posed and examined by Barakat [7]. A complete answer to this question in the form of a necessary and sufficient matrix condition was subsequently presented by the present author [8]. Gil and Bernabeu [9] have recently made the interesting claim that this matrix condition can be replaced by a scalar condition. In the present paper we analyse this claim and show that it is not valid for all situations.

In § 2 we briefly recall the relationship between the descriptions in terms of the coherency matrix and in terms of the Stokes vector leading to the matrix condition [8]. Then we analyse the scalar condition of Gil and Bernabeu and show that it is *not* equivalent to the matrix condition in general. In fact we show that it is only in those situations where a particular Hermitian matrix constructed from the Mueller matrix is positive semidefinite that the scalar condition is equivalent to the matrix condition. In § 3 we present a simple example which illustrates these results. Section 4 contains some concluding remarks.

2. Jones matrix, Mueller matrix and the depolarization criterion

The coherency matrix φ describing a polarization state is a 2×2 complex Hermitian positive semidefinite matrix:

$$\varphi=\begin{bmatrix}\varphi_{11} & \varphi_{12}\\ \varphi_{21} & \varphi_{22}\end{bmatrix},\quad \varphi^\dagger=\varphi,\quad \varphi\geqslant 0. \tag{1}$$

Its transformation by a deterministic (non-image-forming) optical system with Jones matrix J is given by

$$J:\quad \varphi\rightarrow\varphi'=J\varphi J^\dagger. \tag{2}$$

For the purpose of comparison with the Mueller–Stokes formalism it is convenient to associate with every coherency matrix, φ, a four-element column, Φ, in the following one-to-one manner:

$$\Phi=\begin{bmatrix}\varphi_{11}\\ \varphi_{12}\\ \varphi_{21}\\ \varphi_{22}\end{bmatrix}=\begin{bmatrix}\Phi_0\\ \Phi_1\\ \Phi_2\\ \Phi_3\end{bmatrix}. \tag{3}$$

The Stokes vector S describing the same state is related to Φ through a simple numerical matrix A. We have

$$S=A\Phi, \tag{4}$$

where

$$A=\begin{bmatrix}1 & 0 & 0 & 1\\ 1 & 0 & 0 & -1\\ 0 & 1 & 1 & 0\\ 0 & i & -i & 0\end{bmatrix}. \tag{5}$$

It can be easily checked that A is unitary, except for a multiplicative factor, and we have

$$A^{-1}=\tfrac{1}{2}A^\dagger. \tag{6}$$

Since A is non-singular, it follows from (4) that the Stokes vector and the coherency matrix are in one-to-one correspondence, and hence contain identical information about the state of the field. Since φ is Hermitian S is real, and the positive semidefiniteness of φ implies

$$S_0^2-S_1^2-S_2^2-S_3^2\geqslant 0,\quad S_0\geqslant 0 \tag{7}$$

Under the action of an optical element, the change in the polarization state is described through a linear transformation M on S:

$$S \to S' = MS,$$

$$M = \begin{bmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & M_{11} & M_{12} & M_{13} \\ M_{20} & M_{21} & M_{22} & M_{23} \\ M_{30} & M_{31} & M_{32} & M_{33} \end{bmatrix}. \tag{8}$$

The 4×4 real matrix M is called the Mueller matrix of the optical element. In the special case where the optical element under consideration is deterministic it can be described either through a Jones matrix J or a Mueller matrix M, and the two are related through [10, 8]

$$M = A(J \otimes J^*)A^{-1}, \tag{9}$$

where * indicates complex conjugation and $\otimes$ denotes the Kronecker matrix product.

In [8] we defined a matrix N through the elements of M. This is shown as equation (10) on the following page. This relationship between M and N is clearly one-to-one. The matrix N is manifestly Hermitian. Its trace is simply related to M:

$$\mathrm{Tr}(N) = 2M_{00} \tag{11}$$

In the following we will need to use another relationship between M and N:

$$\mathrm{Tr}(N^2) = \mathrm{Tr}(MM^{\mathrm{T}}), \tag{12}$$

where M^{T} denotes the matrix transpose of M. The relationship (12) can easily be verified from the explicit form of N given in (10) by noting that the left-hand side of (12) is the sum of the modulus square of all the 16 elements of N, by virtue of the Hermitian property of N; while the right-hand side is the sum of the squares of the elements of the real matrix M.

In the spirit of (3) we write the 2×2 complex Jones matrix J in (2) as a four-element column vector:

$$J = \begin{bmatrix} J_{11} \\ J_{12} \\ J_{21} \\ J_{22} \end{bmatrix} = \begin{bmatrix} J_0 \\ J_1 \\ J_2 \\ J_3 \end{bmatrix} \tag{13}$$

Even though we use the same symbol J for the 2×2 matrix as for the four-element column, no confusion is expected to arise. For deterministic systems whose Mueller matrix is related to J through (9) the matrix N is related to J in a simple way [8]:

$$N_{\alpha\beta} = J_\alpha J_\beta^*, \quad \alpha, \beta = 0, 1, 2, 3. \tag{14}$$

$$N = \frac{1}{2}\begin{bmatrix} M_{00}+M_{11}+M_{01}+M_{10} & M_{02}+M_{12}+i(M_{03}+M_{13}) & M_{20}+M_{21}-i(M_{30}+M_{31}) & M_{22}+M_{33}+i(M_{23}-M_{32}) \\ M_{02}+M_{12}-i(M_{03}+M_{13}) & M_{00}-M_{11}-M_{01}+M_{10} & M_{22}-M_{33}-i(M_{23}+M_{32}) & M_{20}-M_{21}-i(M_{30}-M_{31}) \\ M_{20}+M_{21}+i(M_{30}+M_{31}) & M_{22}-M_{33}+i(M_{23}+M_{32}) & M_{00}-M_{11}+M_{01}-M_{10} & M_{02}-M_{12}+i(M_{03}-M_{13}) \\ M_{22}+M_{33}-i(M_{23}-M_{32}) & M_{20}-M_{21}+i(M_{30}-M_{31}) & M_{02}-M_{12}-i(M_{03}-M_{13}) & M_{00}+M_{11}-M_{01}-M_{10} \end{bmatrix} \tag{10}$$

Now assume that the optical element is deterministic. Then it has a Jones matrix J, and the N matrix of the system is given by (14). Squaring the matrix equation (14) we have

$$
\begin{aligned}
(N^2)_{\alpha\beta} &= J_\alpha J_\eta^* J_\eta J_\beta^* \\
&= [\mathrm{Tr}(N)] N_{\alpha\beta}.
\end{aligned} \tag{15}
$$

That is, for deterministic systems

$$N^2 = [\mathrm{Tr}(N)] N. \tag{16}$$

Conversely, assume that the optical element satisfies (16). That is, from the Mueller matrix of the given optical element, we construct the N matrix according to the prescription (10), and this matrix satisfies (16). Then (16) implies that $[\mathrm{Tr}(N)]^{-1} N$ is a projection operator, and hence N can be written in the form (14) for some J. In other words, the system can be described through a Jones matrix and hence is deterministic. Thus we have the following theorem [8]: *The necessary and sufficient condition for an optical system with a given Mueller matrix to be deterministic is that its N matrix formed through (10) should satisfy the matrix condition (16).*

Having established (16) we are ready now to analyse the results of other authors in the light of this result. The matrix condition of Barakat will not be analysed here (see [8]). Assume that we have a deterministic system. Then (16) is satisfied. Taking the trace of (16) and using (11) we obtain for such systems

$$\mathrm{Tr}(N^2) = 4 M_{00}^2, \tag{17}$$

and hence from (12),

$$\mathrm{Tr}(M M^{\mathrm{T}}) = 4 M_{00}^2. \tag{18}$$

Thus (18) is a necessary condition for a system to be deterministic. Hence the result of Fry and Kattawar [11] is consistent with out matrix condition. Gil and Bernabeu have claimed that it is also the sufficiency condition. To see if this is so we have to examine whether

$$\mathrm{Tr}(N^2) = [\mathrm{Tr}(N)]^2 \tag{19}$$

is equivalent to (16). Clearly, there are two cases to be distinguished:

Case 1: N is positive semidefinite

In this case it can be seen that (19) is indeed equivalent to (16). This is most easily established by recalling that N is Hermitian, and working in its diagonal representation.

Case 2. N is not positive semidefinite

In this case (16) implies (19); whereas (19) does not imply (16). This too is easily seen in the diagonal representation of N.

Thus the question is reduced to one of whether there exist Mueller matrices whose N matrices will have at least one negative eigenvalue. Such Mueller matrices do indeed exist, and it is precisely for these that the claim of Gil and Bernabeu breaks down. We give examples of such matrices in the following section. But here we note that the scalar condition (18), or equivalently (19), does not replace the matrix condition (16) in view of the fact that the N matrix is not required to be positive definite.

3. Example

As a simple example to illustrate our results in the last section, consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{20}$$

This is a valid Mueller matrix. The conditions (7) mean that the Stokes vector should be a 'time-like' vector with positive time-component. Hence any 4×4 real matrix *which maps every 'time-like' vector with positive time-component into a vector with* these properties is an acceptable Mueller matrix. M in (20) clearly meets this requirement. Formally, these conditions are identical to those imposed on proper Lorentz transformation, but now there is no restriction of invariance of the 'norm' of the vector.

The N matrix corresponding to this M is

$$N = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

It is easily checked that (20) satisfies (18) and, equivalently, (21) satisfies (19). Yet, it does not satisfy the matrix condition (16) and hence does not represent a deterministic system. *In fact there exists no J matrix from which the M can be* derived in the form (9). This simple-looking Mueller matrix which changes neither the intensity nor the degree of polarization of any input state is non-deterministic, for its N matrix is not positive semidefinite; it has eigenvalues $(1, 1, 1, -1)$.

As yet another simple example we cite the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{22}$$

The reader can verify that it represents a non-deterministic system even though it satisfies the scalar condition.

4. Concluding remarks

We have analysed the conditions for a Mueller matrix corresponds to a deterministic system. The necessary and sufficient condition for this is given by the matrix equation (16). In situations in which the eigenvalues of the N matrix formed from the given Mueller matrix are all non-negative, and *only* under these circumstances, is the matrix condition (16) equivalent to the scalar condition (18). Thus the Mueller matrices divide naturally into two disjoint classes: one with positive semidefinite N matrix and the other with an N matrix that has at least one negative eigenvalue. It is of interest to note that a condition similar to (16) is already known in the context of dynamical mapping of the density matrix of a quantum-mechanical system [12].

A related issue of interest is the possibility, or otherwise, of realizing the Mueller matrix of a non-deterministic system as an ensemble of Mueller matrices of deterministic systems. This problem has been examined in a recent paper of Kim, Mandel and Wolf [13].

Acknowledgments

This research was carried out during the author's stay at the University of Rochester, which was supported by the National Science Foundation and by the U.S. Air Force Geophysics Laboratory under AFOSR Task 2310G1. It is a pleasure to acknowledge the hospitality and excellent working conditions provided by Professor Emil Wolf, which made this work possible, and his counsel regarding the presentation of the material in this paper.

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Reprinted from Journal of the Optical Society of America A, Vol. 4, page 433, March 1987

Relationship between Jones and Mueller matrices for random media

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Received March 7, 1986; accepted October 9, 1986

The effect of a linear random medium on the state of polarization of the transmitted light is investigated, and the connection between the Stokes vector formalism and the coherence or polarization matrix formalism is discussed. It is shown that an ensemble of Jones matrices corresponds to the Mueller matrix in general.

1. INTRODUCTION

When light propagates through a linear medium, its polarization properties are usually described either by the Stokes vector formalism[1] or by the coherence matrix (also known as the polarization matrix) that was introduced by Wiener[2] and by Wolf.[3] The effect of many non-image-forming optical devices on the light is then to transform both the Stokes vector and the coherence or polarization matrix, so that the device can be represented by a transformation matrix. This transformation is usually known as the Mueller matrix[4] when it acts on the four-dimensional Stokes vector or as the Jones matrix[5] when it acts on the 2×2 polarization matrix.[6,7]

Even though there exists a one-to-one correspondence between a polarization matrix and a Stokes vector, the description of optical systems in terms of Mueller matrices appears to be applicable to more general situations than does the description in terms of Jones matrices. This was already pointed out by Azzam and Bashara,[8] and Howell[9] has shown that some optical devices can be described by Mueller matrices but not by Jones matrices.

In several recent publications the constraints that must be satisfied for a Mueller matrix to correspond to a Jones matrix were investigated.[10–14] Simon[12] and Barakat[13] found nine constraints that are necessary if a Mueller matrix is to be derivable from a single Jones matrix, and more recently Gil and Bernabeu[14] found a single condition on the trace of the square of the Mueller matrix. These results apply to propagation through a deterministic optical device.

On the other hand, some optical systems are nondeterministic, and they can be represented by an ensemble. In what follows we show that when an ensemble of transformations is introduced to describe certain stochastic non-image-forming optical systems, the two descriptions can be completely reconciled, and both are equally general. In the special case when the ensemble reduces to a single realization, the trace condition of Gil and Bernabeu follows naturally.

In Section 2 we review the properties of polarization matrices and of Stokes vector, and in Section 3 we describe the mathematical transformations that characterize transmission through a deterministic device. In Section 4 we introduce an ensemble of transformations to represent a random linear device, and we examine the corresponding relation between the Jones and Mueller matrices.

2. THE COHERENCE OR POLARIZATION MATRIX AND THE STOKES VECTOR

We consider an optical field in the form of a quasi-monochromatic plane wave propagating in some direction characterized by the unit wave vector $\boldsymbol{\kappa}$, say, the z direction. Let $\mathbf{E}$ be the complex analytic signal representing the transverse vector field. The field can always be resolved into two orthogonal components, 1 and 2:

$$\mathbf{E} = E_1\boldsymbol{\epsilon}_1 + E_2\boldsymbol{\epsilon}_2, \tag{1}$$

where $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2$ are orthogonal unit vectors. $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2$ could be real unit vectors in the x, y directions, corresponding to orthogonal linear polarizations. However, sometimes it is more convenient to resolve the field into more general orthogonal states of elliptic polarization, in which case $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2$ are complex. In any case the transversality of $\mathbf{E}$ is expressed by the condition

$$\boldsymbol{\kappa} \cdot \boldsymbol{\epsilon}_i = 0 \qquad (i = 1, 2) \tag{2}$$

and the orthonormality of $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2$ by

$$\boldsymbol{\epsilon}_i^* \cdot \boldsymbol{\epsilon}_j = \delta_{ij} \qquad (i, j = 1, 2). \tag{3}$$

If the field is fluctuating, then E_1, E_2 in Eq. (1) are random variables described by an ensemble, which we shall assume to be stationary. The 2×2 polarization matrix J is the covariance matrix of the two variates E_1, E_2 and is given by

$$J_{ij} = \langle E_i E_j^* \rangle, \tag{4}$$

where $\langle\,\rangle$ denotes the ensemble average. By definition, J is Hermitian and nonnegative definite, and its trace is a measure of the mean light intensity $\langle \mathbf{E}^* \cdot \mathbf{E} \rangle$. The effect on J of changing from one set to another set of base vectors $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ is describable by a unitary transformation on J. It follows that there always exists a basis $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2$ in which J is diagonal, because every Hermitian matrix can be diagonalized by a unitary transformation.

The degree of polarization P of the light can be expressed either in terms of eigenvalues of J or in terms of the unitary invariants of J in the form[15]

$$P = [1 - 4 \det J/(\mathrm{Tr}\, J)^2]^{1/2}. \tag{5}$$

It follows from either form that $0 \le P \le 1$ and that $P = 1$ when $\det J = 0$ or, equivalently, when one eigenvalue of J is

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zero, so that only one kind of polarization is present. On the other hand, $P = 0$ when the two eigenvalues are equal, and J is then proportional to the unit matrix, corresponding to an equal mixture of both polarization components. It may be shown[15] that any polarization matrix J can be uniquely decomposed into a fully polarized part and a fully unpolarized part.

The four elements of J can also be used to construct four real parameters known as the Stokes parameters,[1] which are given by

$$\begin{aligned} S_0 &= \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle, \\ S_1 &= \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle, \\ S_2 &= \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle, \\ S_3 &= i[\langle E_2 E_1^* \rangle - \langle E_1 E_2^* \rangle] \end{aligned} \tag{6}$$

and also represent the state of polarization of the field. The four parameters are often considered to be the components of a four-vector $\mathbf{S}$, known as the Stokes vector. In terms of the components of $\mathbf{S}$ the degree of polarization is then given by

$$P = (S_1^2 + S_2^2 + S_3^2)^{1/2}/S_0. \tag{7}$$

Another connection between the Stokes parameters S_μ ($\mu = 0, 1, 2, 3$) and the polarization matrix J becomes apparent if we express J as a linear combination of the four linearly independent 2×2 Pauli spin matrices

$$\sigma^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\sigma^{(1)} = \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix},$$

$$\sigma^{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\sigma^{(3)} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \tag{8}$$

which form a complete set for the representation of any 2×2 matrix. Then we find that

$$J_{ij} = \frac{1}{2} S_\mu \sigma_{ij}^{(\mu)} \qquad (\mu = 0, 1, 2, 3), \tag{9}$$

where summation on repeated indices is understood. It follows that the Stokes parameters are simply twice the coefficients in the expansion of the polarization matrix J in terms of Pauli matrices.

This immediately leads to another expression for S_μ. Let us multiply both sides of Eq. (9) on the right by another Pauli spin matrix $\sigma^{(\nu)}$ ($\nu = 0, 1, 2, 3$) and take the trace on both sides. Then we obtain

$$\mathrm{Tr}[J\sigma^{(\nu)}] = \frac{1}{2} S_\mu \mathrm{Tr}[\sigma^{(\mu)}\sigma^{(\nu)}].$$

Recalling that the product of two different Pauli matrices yields one of the three Pauli matrices $\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}$, which is traceless, we see that the only nonvanishing contribution occurs when $\mu = \nu$, in which case the trace equals 2, and we obtain finally

$$S_\mu = \mathrm{Tr}[J\sigma^{(\mu)}] = J_{ij}\sigma_{ji}^{(\mu)} \qquad (\mu = 0, 1, 2, 3), \tag{10}$$

when the summation convention is applied.

3. TRANSMISSION THROUGH A LINEAR OPTICAL SYSTEM

When the light beam is passed through some non-image-forming optical device such that it enters and emerges as a plane wave, the new field vector $\mathbf{E}'$, given by

$$\mathbf{E}' = E_1'\epsilon_1 + E_2'\epsilon_2, \tag{11}$$

has components E_1', E_2' that are often linearly related to the old components E_1, E_2. For brevity we shall henceforth refer to this device as a filter. We may then represent the filter by the 2×2 transformation matrix T, usually known as the Jones matrix,[5] such that

$$E_i' = T_{ij}E_j \qquad (i, j = 1, 2), \tag{12}$$

where summation on repeated indices is again understood. For the moment we take the elements of T to have definite values, i.e., they are not random. Explicit forms of T for certain common filters, such as a compensator or phase plate, a differential absorber, an optical rotator, and a polarizer, have been given.[16,17]

Let us now examine how the polarization matrix J and the Stokes vector $\mathbf{S}$ are affected under this transformation. We find from the definitions [Eqs. (4) and (12)] that the new polarization matrix J' is given by

$$\begin{aligned} J_{ij}' &= \langle E_i' E_j'^* \rangle \\ &= T_{im}\langle E_m E_n^* \rangle T_{jn}^* \\ &= T_{im} J_{mn} T_{nj}^\dagger \end{aligned} \tag{13}$$

or, in matrix form,

$$J' = TJT^\dagger, \tag{14}$$

where $T^\dagger$ is the Hermitian adjoint of T. Hence J' is related to J by a similarity transformation involving the same matrix T that transforms E to E'. However, as we show below, there exist linear filters whose effects are describable not by transformation (12) or (14) but only by an ensemble of such transformations.[8]

Let us now examine the corresponding transformation rule for the Stokes vector $\mathbf{S}$. Under any linear transformation the new Stokes vector $\mathbf{S}'$ is related to the old one by

$$S_\mu' = M_{\mu\nu} S_\nu \qquad (\mu, \nu = 0, 1, 2, 3). \tag{15}$$

The 4×4 transformation matrix $M_{\mu\nu}$ is known as the Mueller matrix.[4] We may readily obtain the form of $M_{\mu\nu}$ when the field vector obeys the transformation Eq. (12) by making use of Eqs. (10) and (14). We then find that

$$\begin{aligned} S_\mu' &= \mathrm{Tr}[J'\sigma^{(\mu)}] \\ &= \mathrm{Tr}[TJT^\dagger\sigma^{(\mu)}] \end{aligned} \tag{16a}$$

or, in component form,

$$S_\mu' = T_{im} J_{mn} T_{jn}^* \sigma_{ji}^{(\mu)}. \tag{16b}$$

We now substitute for J in Eq. (15) from Eq. (9) and obtain

$$S_\mu' = \frac{1}{2}\,\mathrm{Tr}[T\sigma^{(\nu)}T^\dagger\sigma^{(\mu)}]S_\nu. \qquad (17)$$

Comparison with Eq. (15 shows that in this case the Mueller matrix is given by

$$M_{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}[T\sigma^{(\nu)}T^\dagger\sigma^{(\mu)}] = \frac{1}{2}\,\mathrm{Tr}[\sigma^{(\mu)}T\sigma^{(\nu)}T^\dagger] = \frac{1}{2}\,T_{np}T_{qm}{}^\dagger\sigma_{mn}{}^{(\mu)}\sigma_{pq}{}^{(\nu)}, \qquad (18)$$

and it is evidently related to the 2×2 Jones matrix.

It is apparent from Eq. (18) that to every Jones matrix T there corresponds a Mueller matrix M, but the converse is not necessarily true. As we show below, there are physically realizable but nondeterministic linear filters whose effect on the polarization matrix is not expressible in the form of Eq. (14), although the Stokes vector transforms as in Eq. (15). In particular, under the similarity transformation (14), an initially polarized light beam always remains fully polarized, although the degree of polarization of a partially polarized beam can increase or decrease on transmission through a linear filter.

In order to show this let us choose the polarization basis in which the original polarization matrix J is diagonal. If the light is fully polarized, only one eigenvalue, say I_1, is nonzero, and J must be of the form

$$J_{mn} = I_1\delta_{m1}\delta_{n1}. \qquad (19)$$

Needless to say, under these conditions $\det J = 0$, and from Eq. (5) it follows that the degree of polarization $P = 1$. Let us now calculate the degree of polarization P' of the light beam emerging from the linear filter. With the help of Eq. (19) we have, from Eq. (13),

$$J_{ij}' = T_{i1}T_{j1}{}^*I_1$$

so that

$$\det J' = I_1(T_{11}T_{11}{}^*T_{21}T_{21}{}^* - T_{21}T_{11}{}^*T_{11}T_{21}{}^*) = 0. \qquad (20)$$

Hence $P' = 1$, which implies that polarized light remains polarized after passing through any linear filter whose effect is described by Eq. (12) or (14). Evidently, a depolarizing filter is excluded from this category. However, the action of a depolarizing filter on the Stokes vector **S** is still describable by a transformation of the form shown in Eq. (15), although the actual transformation matrix M is then no longer given by Eq. (18). For example, for the fully polarized light described by Eq. (19) the Stokes vector **S** has components $(I_1, 0, 0, I_1)$, whereas the Stokes vector for unpolarized light is always of the form $(I, 0, 0, 0)$. The 4×4 transformation matrix of the form

$$M = \begin{bmatrix} K & a_1 & b_1 & K \\ 0 & a_2 & b_2 & 0 \\ 0 & a_3 & b_3 & 0 \\ 0 & a_4 & b_4 & 0 \end{bmatrix} \qquad (21)$$

converts $(I_1, 0, 0, I_1)$ into $(KI_1, 0, 0, 0)$ and therefore represents a fully depolarizing filter device. The extra degrees of freedom available in M permit this possibility, whereas there is no 2×2 transformation matrix T to represent this filter. In this sense the Mueller transformation matrix appears to be of more general applicability than the Jones matrix.

By using Eq. (18) we may readily derive the simple trace condition on the Mueller matrix that has been shown to apply to any nondepolarizing optical system.[14] If M^T denotes the transpose of M, then from Eq. (18)

$$\mathrm{Tr}(M^TM) = M_{\mu\nu}M_{\mu\nu} = \frac{1}{4}\,\mathrm{Tr}[T\sigma^{(\nu)}T^\dagger\sigma^{(\mu)}]\mathrm{Tr}[T\sigma^{(\nu)}T^\dagger\sigma^{(\mu)}], \qquad (22)$$

where we have made use of the fact that the trace is invariant under cyclic permutation of factors. With the help of the general matrix rules

$$\mathrm{Tr}[A]\mathrm{Tr}[B] = \mathrm{Tr}[A \otimes B] \qquad (23)$$

and

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \qquad (24)$$

where $\otimes$ denotes the direct product, we can reexpress Eq. (22) in the form

$$\begin{aligned}\mathrm{Tr}(M^TM) &= \frac{1}{4}\,\mathrm{Tr}[T\sigma^{(\nu)}T^\dagger\sigma^{(\mu)} \otimes T\sigma^{(\nu)}T^\dagger\sigma^{(\mu)}] \\ &= \frac{1}{4}\,\mathrm{Tr}[(T\sigma^{(\nu)} \otimes T\sigma^{(\nu)})(T^\dagger\sigma^{(\mu)} \otimes T^\dagger\sigma^{(\mu)})] \\ &= \frac{1}{4}\,\mathrm{Tr}[(T \otimes T)(\sigma^{(\nu)} \otimes \sigma^{(\nu)})(T^\dagger \otimes T^\dagger)(\sigma^{(\mu)} \otimes \sigma^{(\mu)})].\end{aligned} \qquad (25)$$

From the explicit form [Eqs. (8)] of the Pauli matrices we find that

$$\frac{1}{2}\,\sigma^{(\nu)} \otimes \sigma^{(\nu)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (26)$$

and this matrix commutes with any 4×4 matrix of the form $T \otimes T$. Moreover, its square is the unit 4×4 matrix. It follows that

$$\mathrm{Tr}(M^TM) = \mathrm{Tr}[(T \otimes T)(T^\dagger \otimes T^\dagger)]$$

and, with the help of Eqs. (24) and (23),

$$\begin{aligned}\mathrm{Tr}(M^TM) &= \mathrm{Tr}(TT^\dagger \otimes TT^\dagger) \\ &= [\mathrm{Tr}(TT^\dagger)]^2.\end{aligned} \qquad (27)$$

But from Eq. (18) we have

$$M_{00} = \tfrac{1}{2}\mathrm{Tr}(TT^\dagger) \qquad (28)$$

so that finally

$$\mathrm{Tr}(MM^\dagger) = 4M_{00}{}^2. \qquad (29)$$

This is the necessary and sufficient condition found by Gil and Bernabeu[14] for a Mueller matrix to represent a nondepolarizing optical system. We see that it holds for every deterministic optical system described by Eq. (12).

4. REPRESENTATION OF A FILTER BY AN ENSEMBLE

So far we have considered only deterministic optical systems. But in some situations, for example when light is passed through the atmosphere, the system is no longer deterministic and must be described by an ensemble of filters.[8] We shall represent a typical element of the ensemble by a 2 × 2 Jones transformation matrix $T^{(e)}$ and assume that it occurs with probability p_e. The action of the ensemble of filters on the incident light **E** is to generate another ensemble of vector fields $\mathbf{E}'^{(e)}$ generally with different polarization states, such that

$$E_i'^{(e)} = T_{ij}^{(e)} E_j. \tag{30}$$

We emphasize that the new ensemble associated with the filter is in addition to the ensemble formed by the various realizations of the incident field. In constructing the elements J_{ij}' of the polarization matrix of the light that has passed through the optical system, we then need to average over the e ensemble also. Thus we obtain from Eqs. (13) and (30)

$$J_{ij}' = \sum_e p_e \langle E_i'^{(e)} E_j'^{(e)*} \rangle$$

$$= \sum_e p_e T_{im}^{(e)} J_{mn} T_{nj}^{(e)\dagger} \tag{31}$$

or

$$J' = \sum_e (p_e T^{(e)} J T^{(e)\dagger}) = \langle T^{(e)} J T^{(e)\dagger} \rangle_e, \tag{32}$$

where $\langle\,\rangle_e$ is a shorthand notation for the average over the e ensemble. This result should be compared with Eq. (14). In this case J' can no longer be related to J by a similarity transformation, as for a single realization of the ensemble.

Moreover, because of the e ensemble, it is no longer true that polarized light passing through the optical system remains fully polarized. Thus, if J is given by Eq. (19) as before, when we calculate the determinant of J' from Eq. (32), we find in place of Eq. (20)

$$\begin{aligned}\det J' = I_1[&\langle T_{11}^{(e)} T_{11}^{(e)*} \rangle_e \langle T_{21}^{(e)} T_{21}^{(e)*} \rangle_e \\ &- \langle T_{21}^{(e)} T_{11}^{(e)*} \rangle_e \langle T_{11}^{(e)} T_{21}^{(e)*} \rangle_e],\end{aligned} \tag{33}$$

where $\langle\,\rangle_e$ again denotes the average over the filter ensemble. In general, $\det J'$ is not zero, so that $P' \neq 1$ and the emerging light is no longer fully polarized. It follows that our ensemble representation of the filter can now accommodate a depolarizing filter also.

Next we calculate the effect of our optical system on the Stokes vector **S**. By going through the same procedure as in the derivation leading to Eqs. (17) and (18), we now find that the Mueller transformation matrix is given by

$$M_{\mu\nu} = \frac{1}{2} \mathrm{Tr}\left[\sum_e p_e \sigma^{(\mu)} T^{(e)} \sigma^{(\nu)} T^{(e)\dagger}\right] \tag{34}$$

or more explicitly in component form by

$$M_{\mu\nu} = \frac{1}{2} \langle T_{np}^{(e)} T_{qm}^{(e)\dagger} \rangle_e \sigma_{mn}^{(\mu)} \sigma_{pq}^{(\nu)}. \tag{35}$$

If we denote the ensemble average of the product of two Jones matrices by

$$\mathcal{T}_{npqm} = \langle T_{np}^{(e)} T_{qm}^{(e)\dagger} \rangle_e, \tag{36}$$

then Eq. (35) becomes

$$M_{\mu\nu} = \frac{1}{2} \mathcal{T}^{npqm} \sigma_{qm}^{(\mu)} \sigma_{pq}^{(\nu)}. \tag{37}$$

When Eq. (18) is rewritten with $\mathcal{T}_{npqm}$ in place of the product, $T_{np}T_{qm}^{\dagger}$ because the ensemble has only a single member, then Eqs. (18) and (37) become formally identical. Similarly, if we make the same substitutions in Eqs. (13) and (31), the two equations become indistinguishable. We have therefore demonstrated a one-to-one correspondence between the transformation laws for the Jones and the Mueller matrices and have shown that $\mathcal{T}_{npqm}$ completely determines both transformations. Any single realization of the Jones matrix $T^{(e)}$ is clearly inadequate to describe the optical system. The ensemble-average product is needed for the calculation of quantities such as J_{ij}' that are of the second order in the field.

Finally, we consider the problem of inverting Eq. (37), or deriving $\mathcal{T}_{npqm}$ from the Mueller matrix $M_{\mu\nu}$. For this purpose we use Eq. (37) to construct the following sum over the indices μ, ν:

$$M_{\mu\nu} \sigma_{ij}^{(\mu)*} \sigma_{kl}^{(\nu)*} = \frac{1}{2} \mathcal{T}_{npqm} \sigma_{mn}^{(\mu)} \sigma_{ij}^{(\mu)*} \sigma_{pq}^{(\nu)} \sigma_{kl}^{(\nu)*}, \tag{38}$$

with summation over repeated indices again understood. But, from definition (8),

$$\sigma_{mn}^{(\mu)} \sigma_{ij}^{(\mu)*} = 2\delta_{mi}\delta_{nj}. \tag{39}$$

When this result is used twice in Eq. (38), we arrive at

$$\frac{1}{2} M_{\mu\nu} \sigma_{ij}^{(\mu)*} \sigma_{kl}^{(\nu)*} = \mathcal{T}_{jkli}, \tag{40}$$

which is the inverse of Eq. (37) and shows explicitly that $\mathcal{T}_{npqm} = \langle T_{np}^{(e)} T_{qm}^{(e)\dagger} \rangle_e$ is completely determined by the Mueller matrix $M_{\mu\nu}$. However, there is no unique procedure for constructing the ensemble of Jones matrices $T_{np}^{(e)}$, except in the degenerate case, when Eq. (29) holds and the ensemble reduces to a single realization.

ACKNOWLEDGMENTS

This research was supported by the National Science Foundation and the U.S. Air Force Geophysics Laboratory under AFOSR Task 2310G1.

E. Wolf is also affiliated with the Institute of Optics, University of Rochester.

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Non-cosmological redshifts of spectral lines

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reprinted from

nature

Reprinted from Nature, Vol. 326, No. 6111, pp. 363-365, 26 March 1987
Macmillan Journals Ltd., 1987

Non-cosmological redshifts of spectral lines

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We showed in a recent report[1] (see also refs 2–4) that the normalized spectrum of light will, in general, change on propagation in free space. We also showed that the normalized spectrum of light emitted by a source of a well-defined class will, however, be the same throughout the far zone if the degree of spectral coherence of the source satisfies a certain scaling law. The usual thermal sources appear to be of this kind. These theoretical predictions were subsequently verified by experiments[5]. Here, we demonstrate that under certain circumstances the modification of the normalized spectrum *of the emitted light* caused by the correlations between the source fluctuations within the source region can produce redshifts of spectral lines in the emitted light. Our results suggest a possible explanation of various puzzling features of the spectra of some stellar objects, particularly quasars.

To explain why source correlations influence the spectrum of the emitted light consider a very simple example. Suppose that two point sources P_1 and P_2 have identical spectra $S_Q(\omega)$ and that measurements on the emitted field are made at some point P. The sources are assumed at rest relative to an observer at P. Assuming that the source fluctuations can be described by a stationary ensemble, the field at P may be characterized by an ensemble $\{V(P,\omega)\}$ of frequency-dependent realizations[6], each of the form

$$V(P,\omega) = Q(P_1,\omega)\frac{e^{ikR_1}}{R_1} + Q(P_2,\omega)\frac{e^{ikR_2}}{R_2} \quad (1)$$

where $\{Q(P_j,\omega)\}$, $(j = 1, 2)$, characterize the strengths of the two fluctuating point sources, R_1 and R_2 are the distances from P_1 to P and from P_2 to P respectively (see Fig. 1) and $k = \omega/c$, c being the speed of light *in vacuo*. For simplicity polarization effects are ignored and hence V and Q are taken to be scalars. The spectrum of the light at P is then given by

$$S_V(P,\omega) = \langle V^*(P,\omega)V(P,\omega)\rangle \quad (2)$$

where the asterisk denotes *the complex conjugate* and the angular brackets denote the ensemble average. On substituting from equation (1) into equation (2) and using the fact that

$$\langle Q^*(P_1,\omega)Q(P_1,\omega)\rangle = \langle Q^*(P_2,\omega)Q(P_2,\omega)\rangle = S_Q(\omega) \quad (3)$$

the following expression is obtained for the spectrum of the emitted light at P:

$$S_V(P,\omega) = \left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)S_Q(\omega) + \left[W_Q(P_1,P_2,\omega)\frac{e^{ik(R_2-R_1)}}{R_1R_2} + \text{c.c.}\right] \quad (4)$$

Here

$$W_Q(P_1,P_2,\omega) = \langle Q^*(P_1,\omega)Q(P_2,\omega)\rangle \quad (5)$$

is the so-called cross-spectral density of the source fluctuations and c.c. denotes the complex conjugate.

The formula (4) shows that the spectrum $S_V(P,\omega)$ is, in general, not just proportional to $S_Q(\omega)$ but is modified by the correlation, characterized by $W_Q(P_1,P_2,\omega)$, between the fluctuations of the two source strengths $Q(P_1,\omega)$ and $Q(P_2,\omega)$. Only in some very special cases, for example, when the source fluctuations are uncorrelated $[W_Q(P_1,P_2,\omega) = 0]$ will $S_V(P,\omega)$ be proportional to $S_Q(\omega)$. Hence, in general, the spectrum of the light generated by two point sources depends not only on their spectra but also on the correlation between the fluctuations of their strengths.

A generalization of the elementary formula (4) for radiation from three-dimensional steady-state (that is, statistically stationary) sources of any state of coherence is known[7]. Of special interest in the present context is the form that the formula takes when the source has the same normalized spectrum $s_Q(\omega)$, $(\int_0^\infty s_Q(\omega)\,d\omega = 1)$ at each point in the source region and has a degree of spectral coherence[3] (appropriately normalized cross-spectral density) $\mu_Q(\mathbf{r}_1,\mathbf{r}_2,\omega)$ that depends on the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of any source points P_1 and P_2 only through their difference $\mathbf{r}_2-\mathbf{r}_1$. If, in addition, for each frequency that significantly contributes to the source spectrum, the spectral correlation length [the effective spatial width $|\Delta\mathbf{r}'|$ of $|\mu(\mathbf{r}',\omega)|$] is small compared to the linear dimensions of the source, the normalized spectrum $s_V^{(\infty)}(\mathbf{u},\omega)$ of the emitted light in the far zone, in a direction specified by a unit vector $\mathbf{u}$, becomes (see equation (3.11) of ref. 8)

$$s_V^{(\infty)}(\mathbf{u},\omega) = \frac{s_Q(\omega)\tilde{\mu}_Q(k\mathbf{u},\omega)}{\int s_Q(\omega)\tilde{\mu}_Q(k\mathbf{u},\omega)\,d\omega} \quad (6)$$

where $\tilde{\mu}_Q(\mathbf{K},\omega)$ is the three-dimensional spatial Fourier transform of the degree of spectral coherence $\mu_Q(\mathbf{r}',\omega) = \mu_Q(\mathbf{r}_2-\mathbf{r}_1,\omega)$.

Let us now choose as the normalized source spectrum $s_Q(\omega)$ a spectral line with a gaussian profile

$$s_Q(\omega) = \frac{1}{\delta\sqrt{2\pi}}\exp[-(\omega-\omega_0)^2/2\delta^2] \quad (\delta \ll \omega_0) \quad (7)$$

and suppose that at each effective frequency ω, the source correlation decreases with the separation $|\mathbf{r}'| = |\mathbf{r}_2-\mathbf{r}_1|$ of any two source points in a gaussian manner, that is

$$\mu_Q(\mathbf{r}',\omega) = \exp[-r'^2/2\sigma_\mu^2(\omega)] \quad (8)$$

On taking the Fourier transform of equation (8) and substituting the resulting expression into equation (6) we obtain the following expression for the normalized spectrum of the emitted light in the far zone (see equation (3.21) of ref. 8)

$$s_V^{(\infty)}(\omega) = \frac{s_Q(\omega)\sigma_\mu^3(\omega)\exp\{-\frac{1}{2}[k\sigma_\mu(\omega)]^2\}}{\int_0^\infty s_Q(\omega)\sigma_\mu^3(\omega)\exp\{-\frac{1}{2}[k\sigma_\mu(\omega)]^2\}\,d\omega} \quad (9)$$

Here, $s_V^{(\infty)}(\omega)$ is written in place of $s_V^{(\infty)}(\mathbf{u},\omega)$, because the spectrum of the far field is now independent of $\mathbf{u}$, as a consequence of the assumed isotropy of μ_Q (see equation (8)).

The formula (9) shows that the spectrum of the emitted light in the far zone depends both on the spectrum of the source fluctuations and on the manner in which the effective source correlation length $\sigma_\mu(\omega)$ depends on the frequency ω.

Let us consider two particular cases. (1) Suppose first that $\sigma_\mu(\omega)$ is independent of ω. Letting ζ denote the (now constant) value of σ_μ and with $s_Q(\omega)$ given by equation (7), one can readily evaluate the integral in the denominator on the right of equation (9) and one then finds that

$$s_V^{(\infty)}(\omega) = \frac{\alpha}{\delta\sqrt{2\pi}}\exp\left[-\left(\omega-\frac{\omega_0}{\alpha^2}\right)^2\Big/2(\delta/\alpha)^2\right] \quad (10)$$

where

$$\alpha^2 = 1+\left(\frac{\delta}{\Delta}\right)^2 \quad (11a)$$

and

$$\frac{1}{\Delta} = \frac{\zeta}{c} \quad (11b)$$

When the source is effectively spatially incoherent, $\zeta \to 0$. Then according to equation (11) $\Delta \to \infty$ and $\alpha \to 1$ and it follows from equations (10) and (7) that in this case

$$s_V^{(\infty)}(\omega) = s_Q(\omega) \quad (12)$$

Hence, in the limiting case of a completely incoherent source of the class that is considered here, the normalized spectrum of the emitted light in the far zone is identical with the normalized

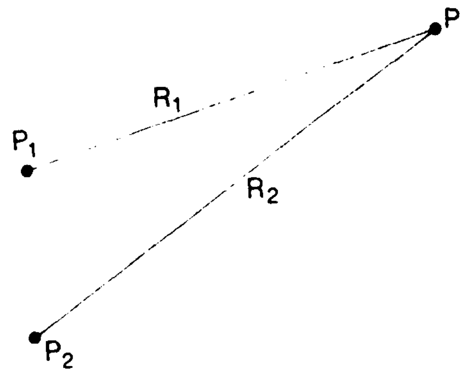

Fig. 1 Illustrating the notation relating to derivation of the formula (4).

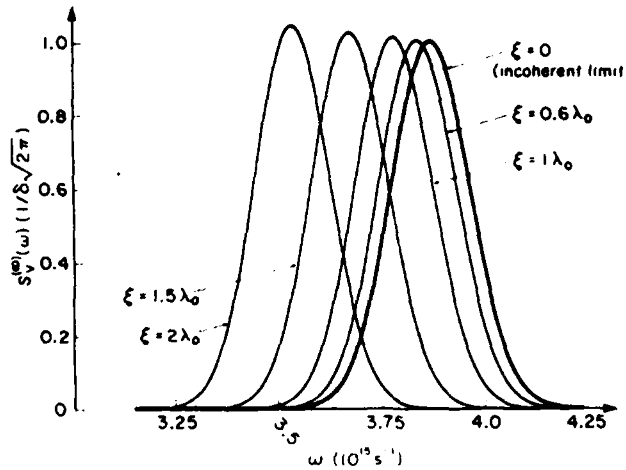

Fig. 2 Spectra $s_V^{(\infty)}(\omega)$ of the far field from sources with spectrum $s_Q(\omega) = (\delta\sqrt{2\pi})^{-1}\exp[-(\omega-\omega_0)^2/2\delta^2]$ and degree of spectral coherence $\mu_Q(\mathbf{r}',\omega) = \exp(-r'^2/2\zeta^2)$, with $\omega_0 = 3.887\times10^{15}\,\mathrm{s}^{-1}$ ($\lambda_0 = 4{,}861$ Å) and $\delta = 9.57\times10^{13}\,\mathrm{s}^{-1}$, for several selected values of the effective source-correlation length ζ. The solid curve ($\zeta\to0$) also represents the source spectrum $s_Q(\omega)$.

spectrum of the source fluctuations.

However, when the source fluctuations are correlated over an effective distance $\zeta > 0$, equation (10) shows that the spectrum $s_V^{(\infty)}(\omega)$, although it is also a line with a gaussian profile, is centred at a lower frequency $\omega_0' = \omega_0/\alpha^2 < \omega_0$. Hence the source correlations give rise to a spectral line $s_V^{(\infty)}(\omega)$ that is redshifted with respect to the spectral line produced by the completely spatially incoherent source with the source spectrum $s_Q(\omega)$. The shifted line is narrower, having root-mean-square width $\delta' = \delta/\alpha < \delta$ and has α-times greater height. Examples of spectra of light in the far zone, produced by several sources which emit the same spectral line but which have different correlation lengths are shown in Fig. 2. From the formula (10) one can readily deduce that the relative shift of the line, namely,

$$z \equiv \frac{\lambda_0 - \lambda_0'}{\lambda_0} = -\frac{\omega_0 - \omega_0'}{\omega_0'} \tag{13}$$

($\lambda_0 = 2\pi c/\omega_0$, $\lambda_0' = 2\pi c/\omega_0'$) is given by

$$z = \left(\frac{\delta}{\Delta}\right)^2 = \left(\frac{\delta}{c}\right)^2 \zeta^2 \tag{14}$$

which shows that in this case the redshift increases quadratically with the spectral source-correlation length ζ. (2) Next consider the situation when $\sigma_\mu(\omega) = a/\omega$ where a is a positive constant. The expression (9) for the normalized spectrum of the emitted light in the far zone now reduces to

$$s_V^{(\infty)}(\omega) = \frac{s_Q(\omega)/\omega^3}{\int_0^\infty [s_Q(\omega)/\omega^3]\,\mathrm{d}\omega} \tag{15}$$

Note that this expression is independent of the value of the constant a.

When $s_Q(\omega)$ is a line with a gaussian profile, given by equation (7), the spectrum $s_V^{(\infty)}(\omega)$, given by equation (15) is no longer strictly gaussian but it can be closely approximated by a gaussian and can be shown to be redshifted with respect to $s_Q(\omega)$ by the relative amount

$$z \approx 3\left(\frac{\delta}{\omega_0}\right)^2. \tag{16}$$

An example of this situation is illustrated in Fig. 3.

This case $[\sigma_\mu(\omega) = a/\omega]$ is of special interest because, according to equation (8), the degree of spectral coherence is now given by

$$\mu_Q(\mathbf{r}',\omega) = \exp[-(kr')^2/2(a/c)^2], \tag{17}$$

that is, it has the functional form

$$\mu_Q(\mathbf{r}',\omega) = f(kr') \quad (k = \omega/c = 2\pi/\lambda) \tag{18}$$

Thus the degree of spectral coherence of the source distribution now satisfies the three dimensional analogue of a requirement (called the scaling law) derived in ref. 1, as a sufficient condition for the spectrum of the light emitted by a planar secondary source of a well-defined class to have certain invariance properties on propagation. It will be shown in another publication (J. T. Foley and E. Wolf, in preparation) that for three-dimensional primary sources of an analogous class, whose degree of coherence satisfies this law, the spectrum of the emitted light has similar invariance properties. We conjecture that the usual thermal sources obey such a scaling law.

Now briefly consider the question of a physical mechanism for producing source correlations. *Such correlations must clearly* be manifestations of some cooperative phenomena. At the atomic level possible candidates may perhaps be superradiance and superfluorescence[9]. An effect of this kind was first predicted by Dicke in 1954 when he showed[10] that under certain circumstances energy from excited atoms may be released cooperatively in a much shorter time than the natural lifetime of the excited states of the atoms and with much larger emission intensity than would be obtained were the atoms radiating independently.

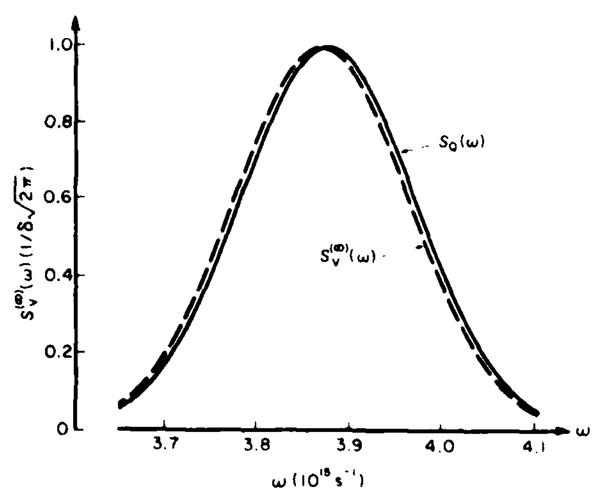

Fig. 3 The spectrum $s_V^{(\infty)}(\omega)$ of the far field from a source with source spectrum $s_Q(\omega) = (\delta\sqrt{2\pi})^{-1}\exp[-(\omega-\omega_0)^2/2\delta^2]$ and degree of spectral coherence $\mu_Q(\mathbf{r}',\omega) = \exp[-(kr')^2/2(a/c)^2]$ (a an arbitrary constant), with $\omega_0 = 3.887\times10^{15}\,\mathrm{s}^{-1}$ ($\lambda_0 = 4{,}861$ Å) and $\delta = 9.57\times10^{13}\,\mathrm{s}^{-1}$. The source spectrum $s_Q(\omega)$ is shown for comparison. Note that $\mu_Q(\mathbf{r}',\omega)$ now obeys the scaling law.

However not enough is known at present about the coherence properties of large three-dimensional systems of this kind to make it possible to determine whether superradiance and superfluorescence might involve correlations that could give rise to spectral line shifts.

There is, however, quite a different mechanism, which can be described at the macroscopic level, and which can imitate effects of source correlations; namely effects of correlations between the refractive index at pairs of points in a spatially random but *statistically homogeneous, time-invariant medium*. If a wave illuminates such a medium, say a dilute gas, then, as is well known, the medium acts as a secondary source, namely as a set of oscillating charges set in motion by the incident wave. The secondary waves produced by the oscillating charges then combine with each other and with the incident wave and generate the scattered field. If the gas is not too dilute the collective response of the microscopic charges to the incident field can be described by macroscopic parameters such as the dielectric susceptibility or the refractive index. Now within the accuracy of the first Born approximation the basic equation for scattering is of the same form as the basic equation for radiation from primary sources, the 'equivalent source' for scattering being the product of the scattering potential (which is a simple function of the refractive index) and of the incident wave. This correspondence clearly implies that our results regarding the effects of source correlations on the spectrum of the emitted light must have analogues regarding the effects of a spatially random medium with correlated refractive index distribution on the spectrum of the light that is scattered by it. This topic will be discussed elsewhere.

Let us now consider some implications of this analysis. Using equation (14), the spectral line in Fig. 2, produced by the source whose correlation length $\zeta = \lambda_0$ is readily found to have a redshift given by $z = 0.0241$ with respect to the source spectrum. It is of interest to note that if an observer detected such a redshift unaware of its true origin and interpreted it on the basis of the Doppler shift formula $v/c = \Delta\lambda/\lambda_0 = z$ he would incorrectly conclude that the source was receeding from him with a speed $v = 0.0241c \approx 7{,}230\ \mathrm{km\ s^{-1}}$.

It seems worthwhile to note that there is a maximum line shift that can be produced by source correlations. This can be seen from the basic formula (6) which indicates that $s_V^{(\infty)}(\mathbf{u}, \omega) = 0$ when $s_Q(\omega) = 0$, implying that the spectrum of the far field can only contain those frequencies that are already present in the source spectrum. Consequently the maximum attainable frequency shift of the line cannot exceed its effective frequency range. However, any frequency contribution from the source spectrum to the normalized spectrum of the far field can be *greatly magnified or greatly reduced*, as is evident from equation (6) and from Fig. 2.

We have mainly considered effects of source correlations under circumstances when the source spectrum consists of a single line and when the degree of spectral coherence μ_Q that characterizes the source correlations depends on a single parameter. Preliminary calculations show that with a suitably chosen μ_Q which depends on a larger number of parameters, redshifts of several lines *may* be produced, all of which will have approximately the same z-values.

In this article we have considered redshifts of spectral lines. However, it is not difficult to specify source correlations which will produce blueshifts. Examples of this kind are given in a forthcoming publication[11].

It seems plausible that the mechanism discussed in this article may be responsible for some of the so far unexplained features of quasar spectra, including line asymmetries and small differences in the observed redshifts of different lines. In this connection it is of interest to recall that the role of coherence in the emission of radiation from quasars was stressed by Hoyle, Burbidge and Sargent in a well-known article[12].

I thank Mr A. Gamliel and Mr K. Kim for carrying out computations relating to the analysis presented in this article. The fact that scattering can also produce shifts of spectral lines was noted independently by Professor Franco Gori, who informed me of this result when commenting on an early version of the manuscript of this article. This investigation was supported by the NSF and by the US Air Force Geophysical Laboratory.

Received 10 September 1986; accepted 3 February 1987

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Red Shifts and Blue Shifts of Spectral Lines Emitted by Two Correlated Sources

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(Received 24 February 1987)

It has recently been shown theoretically that correlations between fluctuations of the source distribution at different source points can produce red shifts or blue shifts of emitted spectral lines. To facilitate experimental demonstration of this effect a simple example is analyzed. It involves only two small appropriately correlated sources and brings out the essential physical features of this new phenomenon.

PACS numbers: 42.68.Hf, 07.65.−b, 42.10.Mg

I showed not long ago that the spectrum of light produced by a fluctuating source depends not only on the source spectrum but also on the correlation that may exist between the source fluctuations at different points within the domain occupied by the source.[1] This result was recently confirmed experimentally.[2] I also showed that under certain circumstances source correlations may produce red shifts or blue shifts of spectral lines in the emitted radiation.[3,4] This prediction has obviously important implications, particularly for astronomy, and it is therefore desirable to verify it also by experiment.

In this Letter I analyze theoretically one of the simplest systems that will generate spectral shifts by this mechanism; namely, two small correlated sources, with identical spectra consisting of a single line of Gaussian profile. I show that with an appropriate choice of the correlation, the spectrum of the emitted radiation will also consist of a single line with a Gaussian profile; however, this emitted line will be red shifted or blue shifted with respect to the spectral line that would be produced if the sources were uncorrelated, the nature of the shift depending on the choice of one of the parameters that specifies the exact form of the correlation coefficient. The main features of this theoretical prediction have been confirmed by Bocko, Douglass, and Knox, using acoustical rather than optical sources. An account of their experiments is given in the accompanying Letter.[5]

Let us consider two small fluctuating sources located at points P_1 and P_2. I assume that the fluctuations are statistically stationary. Let $\{Q(P_1,\omega)\}$ and $\{Q(P_2,\omega)\}$ be the ensembles that represent the source fluctuations[6] at frequency ω. Furthermore, let $\{U(P,\omega)\}$ be the ensemble that represents the field at point P generated by the two sources (Fig. 1). Each realization $U(P,\omega)$ may then be expressed in the form[7]

$$U(P,\omega)=Q(P_1,\omega)\frac{e^{ikR_1}}{R_1}+Q(P_2,\omega)\frac{e^{ikR_2}}{R_2}, \tag{1}$$

where R_1 and R_2 are the distances from P_1 to P and from P_2 to P, respectively, and $k=\omega/c$, c being the speed of light in free space. The spectrum of the field at the point P is given by

$$S_U(P,\omega)=\langle U^*(P,\omega)U(P,\omega)\rangle, \tag{2}$$

where the angular brackets denote ensemble average. On substitution from Eq. (1) into Eq. (2), we find that

$$S_U(P,\omega)=(1/R_1^2+1/R_2^2)S_Q(\omega)+[W_Q(P_1,P_2,\omega)e^{ik(R_2-R_1)}/R_1R_2+\text{c.c.}]. \tag{3}$$

Here

$$S_Q(\omega)=\langle Q^*(P_1,\omega)Q(P_1,\omega)\rangle=\langle Q^*(P_2,\omega)Q(P_2,\omega)\rangle \tag{4}$$

is the spectrum (assumed to be the same) of each of the two source distributions,

$$W_Q(P_1,P_2,\omega)=\langle Q^*(P_1,\omega)Q(P_2,\omega)\rangle \tag{5}$$

is the cross-spectral density of the source fluctuations [first paper of Ref. 6, Eqs. (3.3) and (5.9)], and c.c. denotes the complex conjugate.

The degree of spectral coherence at frequency ω, which is a measure of correlation that may exist between the two fluctuating sources, is given by the formula[8]

$$\mu_Q(P_1,P_2,\omega)=W_Q(P_1,P_2,\omega)/S_Q(\omega). \tag{6}$$

The normalization in Eq. (6) ensures that $0\le|\mu_Q(P_1,P_2,\omega)|\le 1$. The extreme value $|\mu_Q|=1$ characterizes complete correlation (complete spatial coherence) at frequency ω. The other extreme value, $\mu=0$, characterizes complete absence of correlations (complete spatial incoherence).

On substituting for W_Q from Eq. (6) into Eq. (3), we find that

$$S_U(P,\omega)=S_Q(\omega)\{1/R_1^2+1/R_2^2+[\mu_Q(\omega)e^{ik(R_2-R_1)}/R_1R_2+\text{c.c.}]\}, \tag{7}$$

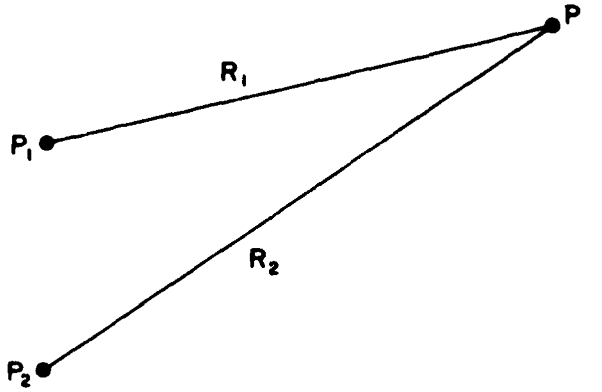

FIG. 1. Geometry and notation relating to the determination of the spectrum $S_U(P,\omega)$ of the field at P produced by two small sources with identical spectra $S_Q(\omega)$ located at P_1 and P_2.

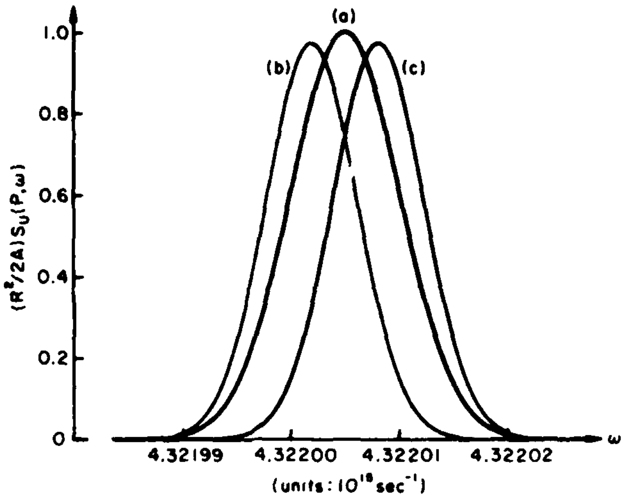

FIG. 2. Red shifts and blue shifts of spectral lines as predicted by formula (12). The spectrum $S_Q(\omega)$ of each of the two source distributions is a line with a Gaussian profile given by Eq. (9) with $A=1$, $\omega_0=4.32201\times10^{15}$ sec^{-1} (Hg line $\lambda=4358.33$ Å), $\delta_0=5\times10^9$ sec^{-1}. (a) The field spectrum $S_U(P,\omega)$ at P when the two sources are uncorrelated ($\mu_Q\equiv0$). (b),(c) The field spectra at P when the two sources are correlated in accordance with Eq. (10), with $a=1.8$, $\delta_1=7.5\times10^9$ sec^{-1}, and (b) $\omega_1=\omega_0-2\delta_0$ (red-shifted line), (c) $\omega_1=\omega_0+2\delta_0$ (blue-shifted line).

where I have omitted the arguments P_1 and P_2 in μ. For the sake of simplicity, let us choose the field point P to lie on the perpendicular bisector of the line joining P_1 and P_2. Then $R_1=R_2$ ($=R$, say) and formula (7) reduces to

$$S_U(P,\omega)=(2/R^2)S_Q(\omega)[1+\mathrm{Re}\,\mu_Q(\omega)], \tag{8}$$

where Re denotes the real part.

We note in passing that when either $\mu_Q(\omega)\equiv0$ (mutually completely uncorrelated sources) or when $\mu_Q(\omega)\equiv1$ (mutually completely correlated sources), the spectrum $S_U(P,\omega)$ of the field at the point P will be proportional to the spectrum $S_Q(\omega)$ of the source fluctuations. However, in general this will not be the case. In fact, it is clear from formula (8) that the field spectrum may differ drastically from the source spectrum, the difference depending on the behavior of the correlation coefficient $\mu_Q(\omega)$ as a function of frequency.

Suppose now that the spectrum of each of the two sources consists of a single line of the same Gaussian profile,

$$S_Q(\omega)=A\,e^{-(\omega-\omega_0)^2/2\delta_0^2}, \tag{9}$$

where A, ω_0, and δ_0 ($\ll\omega_0$) are positive constants. Suppose further that the correlation between the two sources is characterized by the degree of spectral coherence

$$\mu_Q(\omega)=a\,e^{-(\omega-\omega_1)^2/2\delta_1^2}-1, \tag{10}$$

where a, ω, and δ ($\ll\omega_1$) are also positive constants. In order that expression (10) is a degree of spectral coherence, I must also demand that $a\le2$. On substituting from Eqs. (9) and (10) into Eq. (8), I obtain the following expression for the spectrum of the field at the point P:

$$S_U(P,\omega)=\frac{2Aa}{R^2}e^{-(\omega-\omega_0)^2/2\delta_0^2}e^{-(\omega-\omega_1)^2/2\delta_1^2}. \tag{11}$$

By straightforward calculation one can show that this expression may be rewritten in the form

$$S_U(P,\omega)=A'e^{-(\omega-\omega_0')^2/2\delta_0'^2}, \tag{12}$$

where

$$A'=(2Aa/R^2)e^{-(\omega_1-\omega_0)^2/2(\delta_0^2+\delta_1^2)}, \tag{13}$$

$$\omega_0'=(\delta_1^2\omega_0+\delta_0^2\omega_1)/(\delta_0^2+\delta_1^2), \tag{14}$$

and

$$1/\delta_0'^2=1/\delta_0^2+1/\delta_1^2. \tag{15}$$

On the other hand, were the two sources uncorrelated, the correlation coefficient μ_Q would have zero value and we would then have, according to Eqs. (8) and (9),

$$[S_U(P,\omega)]_{\mathrm{uncorr}}=(2A/R^2)e^{-(\omega-\omega_0)^2/2\delta_0^2}. \tag{16}$$

Comparison of Eq. (12) with Eq. (16) shows that although both the spectral lines have Gaussian profiles, they differ from each other. Since according to Eq. (15) $\delta_0'<\delta_0$, the spectral line from the correlated sources is narrower than the spectral line from the uncorrelated sources. Further, we can readily deduce from Eq. (14) that

$$\omega_0'\lesseqgtr\omega_0$$

according as

$$\omega_1 \lessgtr \omega_0.$$

Hence if $\omega_1 < \omega_0$ the spectral line (12) produced by the correlated sources is centered on a lower frequency than the spectral line (16) from two uncorrelated sources, i.e., it is *red shifted* with respect to it; and if $\omega_1 > \omega_0$ the spectral line (12) is *blue shifted* with respect to the spectral line (16). Figure 2 illustrates these results by simple examples.

The preceding considerations show clearly the possibility of generating, by means of correlations between source fluctuations, either red shifts or blue shifts of lines in the spectrum of radiation emitted by sources that are stationary with respect to an observer.

I am obliged to Mr. A. Gamliel for carrying out the computations relating to Fig. 2. This research was supported by the U.S. National Science Foundation and by the U.S. Air Force Geophysics Laboratory under Air Force Office of Scientific Research Task No. 2310G1.

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THE RADIANCE AND PHASE-SPACE REPRESENTATIONS OF THE CROSS-SPECTRAL DENSITY OPERATOR ☆

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Received 17 October 1986

Hilbert space operators are introduced into classical wave theory, which make it possible to associate a unique operator with the cross-spectral density. By linearly mapping this operator onto an associated phase-space one obtains a wide class of generalized radiance functions, including two well-known ones that were introduced by Walther in a different manner. When the source is quasi-homogeneous and the wavelength is short enough all these functions become identical; and this unique limit is found to have all the properties of the traditional radiance, at least in the source plane.

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1. Introduction

In order to clarify the foundations of radiometry a number of authors proposed various expressions for the radiance, in terms of the cross-spectral density of the light distribution across the source [1-5]. Unfortunately none of them satisfies all the postulates of radiometry for any state of coherence of the light and it is now know that none in fact exists, if the radiance is to be linearly related to the cross-spectral density [6]. Very recently it was shown, however, that if the source is quasi-homogeneous, the expressions for the radiance proposed by Walther [4,1] acquire, in the limit of short wavelengths, all the properties that one postulates for it in traditional radiometry [7,8].

It was noted [9] that in its mathematical structure radiometry has much in common with the phase-space representation of quantum mechanics. In particular the phase-space representation of quantum mechanics deals with functions which are c-number representatives of pairs of conjugate operators and the radiance is a function of a pair of variables that are conjugate in the sense of Fourier theory. In the present paper we investigate this similarity further and we show that it leads to a clarification of the true significance of the radiance.

We show in sec. 2 that even within the framework of classical wave theory one may introduce non-commuting operators [#1] $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{s}}_{\perp}$ which are associated with position ($\boldsymbol{\rho}$) and direction ($\boldsymbol{s}_{\perp}$) respectively. In sec. 3 we associate a unique Hilbert space operator $\hat{G} \equiv G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_{\perp})$ with the cross-spectral density. Using this fact we then briefly indicate how a whole class of generalized radiance functions $\mathscr{B}_{\Omega}(\boldsymbol{\rho}, \boldsymbol{s})$ may be introduced by linearly mapping $G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_{\perp})$ onto an associated $\boldsymbol{\rho}, \boldsymbol{s}_{\perp}$-phase space. In sec. 4 we give explicit expression for such mappings and we illustrate the results by showing that the two expressions for radiance proposed by Walther are (apart from a trivial factor) just the phase space representatives of $G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_{\perp})$ obtained according to the so-called Weyl rule and the antistandard rule of mapping (operator ordering). In the concluding section (sec. 5) we show that in the short wavelength limit all the generalized radiance functions become identical. Finally we show that when the source is quasi-homogeneous, this unique limit is a function that has all the properties

☆ Research supported by the National Science Foundation and by the Air Force Geophysics Laboratory under AFOSR Task 2310G1.

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[#1] We denote operators with a caret.

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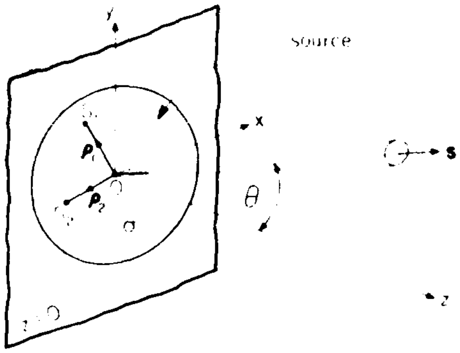

$s^2 = 1$, $s = (s_x, s_y, s_z)$, $s_\perp = (s_x, s_y, 0)$, $s_z = \cos\theta$

Fig. 1. Notation relating to formula (2.1) for the radiant intensity.

postulated for the radiance in traditional radiometry, at least in the source plane.

2. Position operator, its conjugate operator and commutation relations for classical wave theory

Consider a steady-state (i.e. statistically stationary), planar, secondary source, occupying a portion σ of the plane $z=0$ and radiating into the half-space $z>0$. The radiant intensity $J_\nu(\boldsymbol{s})$, i.e. the rate at which the source radiates energy at frequency ν per unit solid angle around a direction specified by a real unit vector $\boldsymbol{s}$, is known to be given by the expression [10]

$$J_\nu(\boldsymbol{s}) = (2\pi k s_z)^2\, \tilde{W}(k\boldsymbol{s}_\perp, -k\boldsymbol{s}_\perp, \nu)\,. \tag{2.1}$$

Here

$$\tilde{W}(\boldsymbol{f}_1, \boldsymbol{f}_2, \nu) = (2\pi)^{-4} \iint W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu)$$

$$\times \exp[-i(\boldsymbol{f}_1\cdot\boldsymbol{\rho}_1 + \boldsymbol{f}_2\cdot\boldsymbol{\rho}_2)]\, d^2\rho_1\, d^2\rho_2 \tag{2.2}$$

is the four-dimensional spatial Fourier transform of the cross-spectral density function $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu)$. Further $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are position vectors of any two source points S_1 and S_2, $\boldsymbol{s}_\perp \equiv (s_x, s_y, 0)$ is the component (considered as a two-dimensional vector) of $\boldsymbol{s}$ parallel to the source plane and s_z is the component of $\boldsymbol{s}$ along the normal to the source plane (see fig. 1).

In the domain of physical optics, the radiant intensity $J_\nu(\boldsymbol{s})$ is the main measurable quantity relating to radiation generated by the source. In radiometry the chief quantity is the radiance $B_\nu(\boldsymbol{\rho}, \boldsymbol{s})$ which is regarded as representing the rate at which energy at frequency ν is radiated from a source element of unit area at $\boldsymbol{\rho}$ into a unit solid angle around the $\boldsymbol{s}$-direction. A basic radiometric formula, which is always introduced from intuitive geometrical considerations, expresses the radiant intensity in terms of the radiance as

$$J_\nu(\boldsymbol{s}) = s_z \int_\sigma B_\nu(\boldsymbol{\rho}, \boldsymbol{s})\, d^2\rho\,. \tag{2.3}$$

In attempts to clarify the relation between radiometry and classical wave theory various expressions have been proposed for the radiance in terms of the cross-spectral density ([1,5]). For reasons that will become apparent shortly we will refer to the quantities introduced in this way as *generalized radiance functions* and denote them by the symbol $\mathcal{B}_\nu(\boldsymbol{\rho}, \boldsymbol{s})$. When appropriate we will attach a superscript to this symbol to distinguish between the different definitions. As already mentioned in the introduction, the various expressions proposed for the radiance have very similar mathematical structure as some of the phase-space representatives of quantum-mechanical operators. This similarity suggests that the generalized radiance functions may be just different phase-space representatives of one and the same Hilbert space operator. We will show later that this indeed is the case. Before doing so we will introduce a set of non-commuting operators into classical wave theory.

We consider the Hilbert space of square-integrable functions of $\boldsymbol{\rho}$. We associate with each cartesian coordinate x and y of $\boldsymbol{\rho}$ operators $\hat{x}$ and $\hat{y}$ whose eigenstates $|x\rangle$ and $|y\rangle$ are defined in the usual way:

$$\hat{x}|x\rangle = x|x\rangle\,, \quad \hat{y}|y\rangle = y|y\rangle\,. \tag{2.4}$$

Next we introduce the vector operator $\hat{\boldsymbol{\rho}} = (\hat{x}, \hat{y})$ and the tensor product eigenstates $|\boldsymbol{\rho}\rangle = |x\rangle|y\rangle$. It follows from eqs. (2.4) that $\hat{\boldsymbol{\rho}}|\boldsymbol{\rho}\rangle = \boldsymbol{\rho}|\boldsymbol{\rho}\rangle$.

For our purposes the variables conjugate to x and y and the associated operators may be most naturally introduced in the following way. We consider monochromatic wavefields $U(\boldsymbol{r}, \nu)\exp(-2\pi i\nu t)$ in the half-space $z>0$, which behave as outgoing spherical waves at infinity in that half-space. $U(\boldsymbol{r}, \nu)$ then satisfies, throughout that half-space, the Helmholtz equation

$$(\nabla^2 + k^2)\,U(\boldsymbol{r}, \nu) = 0\,, \tag{2.5}$$

where

$$k = 2\pi\nu/c\,, \tag{2.6}$$

c being the speed of light in vacuo. It is well known that under very general conditions $U(\boldsymbol{r}, \nu)$ may be expressed in the form of an angular spectrum of plane waves, viz., [11]

$$U(\boldsymbol{r}, \nu) = \int a(\boldsymbol{s}_\perp, \nu) \exp(\mathrm{i}k\boldsymbol{s}\cdot\boldsymbol{r})\, \mathrm{d}^2 s_\perp\,. \tag{2.7}$$

Here $\boldsymbol{s} \equiv (s_x, s_y, s_z) \equiv (\boldsymbol{s}_\perp, s_z)$ is again a unit vector but it may now take on complex values. More precisely, s_x and s_y with $0 \le s_x < \infty$, $0 \le s_y < \infty$ are real and

$$s_z = (1 - s_x^2 - s_y^2)^{1/2} \quad \text{when } s_x^2 + s_y^2 \le 1\,, \tag{2.8a}$$

$$= \mathrm{i}(s_x^2 + s_y^2 - 1)^{1/2} \quad \text{when } s_x^2 + s_y^2 > 1\,. \tag{2.8b}$$

Let us now specialize eq. (2.7) to the representation of the field at points $\boldsymbol{\rho} = (x, y, 0)$ in the source plane $z = 0$ and let us operate on it with the transverse laplacian, $\nabla_\perp \equiv (\partial/\partial x, \partial/\partial y)$. We then find that

$$-\mathrm{i}\lambda\!\!\!^{-}\nabla_\perp U(\boldsymbol{\rho}, \nu) = \int \boldsymbol{s}_\perp a(\boldsymbol{s}_\perp, \nu) \exp(\mathrm{i}k\boldsymbol{s}_\perp\cdot\boldsymbol{\rho})\, \mathrm{d}^2 s_\perp\,, \tag{2.9}$$

where

$$\lambda\!\!\!^{-} = \lambda/2\pi = 1/k = c/2\pi\nu \tag{2.10}$$

is the "reduced" wavelength. Eq. (2.9) suggests that we associate with $\boldsymbol{s}_\perp$ an operator $\hat{\boldsymbol{s}}_\perp$ by the formula

$$\hat{\boldsymbol{s}}_\perp = -\mathrm{i}\lambda\!\!\!^{-}\nabla_\perp\,. \tag{2.11}$$

Its cartesian components

$$\hat{s}_x = -\mathrm{i}\lambda\!\!\!^{-}\partial/\partial x\,, \quad \hat{s}_y = -\mathrm{i}\lambda\!\!\!^{-}\partial/\partial y\,, \tag{2.12}$$

together with the position operators $\hat{x}$ and $\hat{y}$, may readily be shown to obey the commutation relations

$$[\hat{x}, \hat{s}_x] = \mathrm{i}\lambda\!\!\!^{-}\,, \quad [\hat{y}, \hat{s}_y] = \mathrm{i}\lambda\!\!\!^{-}\,, \tag{2.13}$$

which are analogous to the quantum mechanical communitation relations for position and momentum [12].

3. A class of generalized radiance functions

In an important paper dealing with radiometry and coherence, Walther [1] introduced the generalized radiance function

$$\mathscr{B}_\nu^{(\mathrm{W})}(\boldsymbol{\rho}, \boldsymbol{s}) = (k/2\pi)^2 s_z F_\nu^{(\mathrm{W})}(\boldsymbol{\rho}, \boldsymbol{s}_\perp)\,, \tag{3.1}$$

where

$$F_\nu^{(\mathrm{W})}(\boldsymbol{\rho}, \boldsymbol{s}_\perp) = \int W(\boldsymbol{\rho} + \tfrac{1}{2}\boldsymbol{\rho}', \boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}', \nu)$$

$$\times \exp(-\mathrm{i}k\boldsymbol{s}_\perp\cdot\boldsymbol{\rho}')\, \mathrm{d}^2\rho'\,. \tag{3.2}$$

In these formulas $\boldsymbol{s}$ denotes a real unit vector and $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu)$ represents the cross-spectral density in the source plane.

In the phase-space representation of quantum mechanics a function $F^{(\mathrm{W})}$ of the form given by the integral (3.2) is the so-called Wigner representative of an operator $\hat{G}$ that depends on a pair of non-commuting variables [13]. In analogy with the relation between the Wigner distribution function $F^{(\mathrm{W})}$ and the operator $\hat{G}$ which it represents in phase space, we will rewrite eq. (3.2) in the form [2]

$$F_\nu^{(\mathrm{W})}(\boldsymbol{\rho}, \boldsymbol{s}_\perp) = \int \langle \boldsymbol{\rho} + \tfrac{1}{2}\boldsymbol{\rho}' | G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_\perp) | \boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}' \rangle$$

$$\times \exp(-\mathrm{i}k\boldsymbol{s}_\perp\cdot\boldsymbol{\rho}')\, \mathrm{d}^2\rho'\,, \tag{3.3}$$

where [3]

$$\langle \boldsymbol{\rho}_1 | G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_\perp) | \boldsymbol{\rho}_2 \rangle = W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu)\,. \tag{3.4}$$

The function $F_\nu^{(\mathrm{W})}(\boldsymbol{\rho}, \boldsymbol{s}_\perp)$, defined by these two equations, may be said to be the Wigner representative of the Hilbert space operator $\hat{G} = G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_\perp)$. Eq. (3.4) shows that the matrix elements of this operator are just the appropriate values of the cross-spectral density $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu)$. It seems worthwhile to stress that in spite of its close resemblance to the phase space representation of quantum mechanics the above representation is based entirely on classical theory.

The operator $\hat{G} = G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_\perp)$ that we just introduced

[2] Since the present theory is based on classical rather than quantum theory, $k = 1/\lambda\!\!\!^{-}$ rather $1/\hbar$ ($\hbar = h/2\pi$, h being Planck's constant) appears in eq. (3.3).

[3] The operator $\hat{G} = G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_\perp)$ and also the operator $A(\boldsymbol{\rho} \to \hat{\boldsymbol{\rho}}, \boldsymbol{s}_\perp \to \hat{\boldsymbol{s}}_\perp)$ and the function $\Omega(\boldsymbol{u}, \boldsymbol{v})$ introduced below depend on ν, but we do not display this dependence.

via eq. (3.4) may be expressed in many different ways by the use of the commutation relations (2.13). More specifically, if $\hat{G}$ is expressed in the form of a power series in the cartesian components of $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{s}}_{\perp}$, each term involving a product of these elementary operators may be arranged according to some chosen rule of ordering (cf. [14]). One can associate with each such ordered series a c-number representative (phase-space representative) of $\hat{G}$. The function $F_{\nu}^{(W)}$, defined by eq. (3.3), is among the best known representations of the operator. It may be shown to be associated with $\hat{G}$ via the so-called Weyl rule of ordering.

It is evident that other generalized radiance functions can be introduced via formulas of the form (3.1) and (3.2), with $F_{\nu}^{(W)}(\boldsymbol{\rho}, \boldsymbol{s}_{\perp})$ replaced by other phase-space representatives of the operator $\hat{G}$. If we label the different representatives of $\hat{G}$ by superscript Ω we will then have in place of eq. (3.1) the formula

$$\mathscr{B}_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s}) = (k/2\pi)^2 s_z F_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s}_{\perp}) . \tag{3.5}$$

We will consider phase-space representatives $F_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s}_{\perp})$ produced by mappings of the class investigated in ref. [14]. Each $F_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s}_{\perp})$ is then related linearly to $G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_{\perp})$ and it follows from eqs. (3.4) and (3.5) that the associated generalized radiance function

$$\mathscr{B}_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s}) = L^{(\Omega)}\{W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu)\} , \tag{3.6}$$

where $L^{(\Omega)}$ denotes a linear transformation.

We will impose on $\mathscr{B}_{\nu}^{(\Omega)}$ the constraint that

$$J_{\nu}(\boldsymbol{s}) = s_z \int \mathscr{B}_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s})\, d^2\rho , \tag{3.7}$$

where $J_{\nu}(\boldsymbol{s})$ is the radiant intensity given by the expression (2.1) of physical optics. In eq. (3.7) the integral extends over the whole source plane $z = 0$. It will, of course, reduce to the radiometric expression (2.3) when $\mathscr{B}_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s})$ vanishes for all $\boldsymbol{\rho}$-vectors that specify points in the source plane outside the region σ occupied by the source.

4. Expressions for the generalized radiance functions

It is known form the general theory of phase-space representations of functions on non-commuting operators [14] that every linear mapping (of a broad well-defined class) of operator functions $G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_{\perp})$ onto c-number functions $F_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s}_{\perp})$ is characterized by a filter function $\Omega(\boldsymbol{u}, \boldsymbol{v})$, with the following properties:

(a) $\Omega(\boldsymbol{u}, \boldsymbol{v})$ is an entire analytic function of the four complex variables $\boldsymbol{u} = (u_x, u_y)$, $\boldsymbol{v} = (v_x, v_y)$,

(b) $\Omega(\boldsymbol{u}, \boldsymbol{v})$ has no zeros on the real u_x-, u_y-, v_x- and v_y-axes,

(c) $\Omega(0, 0) = 1$.

The explicit expression for $F_{\nu}^{(\Omega)}$ in terms of $\hat{G}$ is [a]

$$F_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s}_{\perp}) = (2\pi\lambda)^2 \int \langle \boldsymbol{\rho}_1 | G(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{s}}_{\perp})$$

$$\times \Delta^{(\tilde{\Omega})}(\boldsymbol{\rho} - \hat{\boldsymbol{\rho}}, \boldsymbol{s}_{\perp} - \hat{\boldsymbol{s}}_{\perp}) | \boldsymbol{\rho}_1 \rangle\, d^2\rho_1 , \tag{4.1}$$

where

$$\Delta^{(\tilde{\Omega})}(\boldsymbol{\rho} - \hat{\boldsymbol{\rho}}, \boldsymbol{s}_{\perp} - \hat{\boldsymbol{s}}_{\perp}) = (2\pi)^{-4} \iint \tilde{\Omega}(\boldsymbol{u}, \boldsymbol{v})$$

$$\times \exp\{-i[\boldsymbol{u}\cdot(\boldsymbol{\rho} - \hat{\boldsymbol{\rho}}) + \boldsymbol{v}\cdot(\boldsymbol{s}_{\perp} - \hat{\boldsymbol{s}}_{\perp})]\}\, d^2u\, d^2v , \tag{4.2}$$

and

$$\tilde{\Omega}(\boldsymbol{u}, \boldsymbol{v}) = [\Omega(-\boldsymbol{u}, -\boldsymbol{v})]^{-1} . \tag{4.3}$$

On substituting from eq. (4.2) into eq. (4.1) and then substituting the resulting expression for $F_{\nu}^{(\Omega)}$ into eq. (3.5) and making use of eq. (3.4) we obtain, after some calculation, the following expression for $\mathscr{B}_{\nu}^{(\Omega)}$:

$$\mathscr{B}_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s}) = (2\pi)^{-4} s_z \iiint \tilde{\Omega}(\boldsymbol{u}, \boldsymbol{v})$$

$$\times \exp[-i(\boldsymbol{u}\cdot\boldsymbol{\rho} + \boldsymbol{v}\cdot\boldsymbol{s}_{\perp} + \tfrac{1}{2}\lambda\boldsymbol{u}\cdot\boldsymbol{v})]\, W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1 - \lambda\boldsymbol{v}, \nu)$$

$$\times \exp(i\boldsymbol{u}\cdot\boldsymbol{\rho}_1)\, d^2u\, d^2v\, d^2\rho_1 . \tag{4.4}$$

We see that $\mathscr{B}_{\nu}^{(\Omega)}$ is indeed a linear transform of the cross-spectral density W [see eq. (3.6)].

Now $\mathscr{B}_{\nu}^{(\Omega)}$ must also satisfy the relation (3.7), with the radiant intensity $J_{\nu}(\boldsymbol{s})$ given by eq. (2.1), i.e. it must satisfy the relation

$$\int \mathscr{B}_{\nu}^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s})\, d^2\rho = (2\pi)^2 s_z \tilde{W}(k\boldsymbol{s}_{\perp}, -k\boldsymbol{s}_{\perp}, \nu) . \tag{4.5}$$

[a] Many of the formulas pertaining to the mapping theory developed in ref. [14] contain the "reduced" Planck's constant $\hbar$, but they also apply to the present case if one replaces $\hbar$ by λ

This requirement places a certain constraint on the admissible filter functions $\Omega(\boldsymbol{u}, \boldsymbol{v})$. It is a straightforward matter to show that the constraint is

$$\hat{\Omega}(0, \boldsymbol{v}) = 1 \quad \text{for all } \boldsymbol{v} . \tag{4.6}$$

Let us now consider some examples. For the *Weyl rule* of ordering ([14], sec. VII) $\Omega = \Omega^{(\mathrm{W})}$ where

$$\hat{\Omega}^{(\mathrm{W})}(\boldsymbol{u}, \boldsymbol{v}) = 1 . \tag{4.7}$$

If we use this fact in the general formula (4.4) we find, after changing one of the variables of integration from $\boldsymbol{v}$ to $\boldsymbol{\rho}' = \boldsymbol{v}/\lambda$ that $\mathscr{B}_\nu = \mathscr{B}_\nu^{(\mathrm{W})}$, where

$$\mathscr{B}_\nu^{(\mathrm{W})}(\boldsymbol{\rho}, \boldsymbol{s}) = (k/2\pi)^2 s_z \int W(\boldsymbol{\rho} + \tfrac{1}{2}\boldsymbol{\rho}', \boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}', \nu)$$

$$\times \exp(-\mathrm{i}k\boldsymbol{s}_\perp \cdot \boldsymbol{\rho}')\, \mathrm{d}^2\rho' . \tag{4.8}$$

The expression on the right is precisely the first expression proposed by Walther [1] for the radiance function, which we have already encountered [eqs. (3.1) and (3.2) above].

For the so-called *antistandard rule* ([14], sec. VII) of ordering, $\Omega = \Omega^{(\mathrm{AS})}$ where

$$\hat{\Omega}^{(\mathrm{AS})}(\boldsymbol{u}, \boldsymbol{v}) = \exp(-\mathrm{i}\lambda \boldsymbol{u}\cdot\boldsymbol{v}/2) . \tag{4.9}$$

If we use this expression in the general formula (4.4) we find that

$$\mathscr{B}_\nu^{(\mathrm{AS})}(\boldsymbol{\rho}, \boldsymbol{s}) = (k/2\pi)^2 s_z \exp(\mathrm{i}k\boldsymbol{s}_\perp \cdot \boldsymbol{\rho})$$

$$\times \int W(\boldsymbol{\rho}_1, \boldsymbol{\rho}, \nu) \exp(-\mathrm{i}k\boldsymbol{s}_\perp \cdot \boldsymbol{\rho}_1)\, \mathrm{d}^2\rho_1 . \tag{4.10}$$

The expression on the right is the complex form of the second expression proposed by Walther [4] for the radiance function.

5. The short wavelength limit with quasi-homogeneous sources

Although the procedure outlined in the previous sections leads to a large class of generalized radiance functions, it is clear from the remarks made in the introduction that none of them will satisfy all the postulates of traditional radiometry for sources of any state of coherence. However, as we will now show, our theory leads to a very general result regarding the foundations of radiometry.

We have seen that the different phase-space representatives $F^{(\Omega)}$ of $\hat{G}$ and consequently the various generalized radiance functions $\mathscr{B}_\nu^{(\Omega)}$ are associated with different rules of ordering of products involving the noncommuting operators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{s}}_\perp$. However, it is seen from eq. (2.13) that in the limit as $\lambda \to 0$, these operators will commute and the distinction between the different types of ordering will then disappear. Consequently all the phase-space representatives $F^{(\Omega)}$ of the operator $\hat{G}$ and hence also *all the generalized radiance functions $\mathscr{B}_\nu^{(\Omega)}$ will become identical in the short-wavelength limit*. However, in view of Friberg's theorem [6] this limiting expression cannot be expected to have all the properties attributed to the radiance function in traditional radiometry for sources of any state of coherence.

It was recently shown [7] that when the source is quasi-homogeneous the generalized radiance function (4.10) is in the limit as $\lambda \to 0$ (more precisely in the asymptotic limit as $k = 1/\lambda \to 0$) given by the expression

$$\mathscr{B}_\nu(\boldsymbol{\rho}, \boldsymbol{s}) = k^2 s_z I^{(0)}(\boldsymbol{\rho}, \nu)\, \tilde{g}^{(0)}(k\boldsymbol{s}_\perp, \nu) \quad \text{when } \boldsymbol{\rho} \in \sigma$$

$$= 0 \quad \text{when } \boldsymbol{\rho} \notin \sigma , \tag{5.1}$$

where $I^{(0)}(\boldsymbol{\rho}, \nu)$ represents the intensity distribution across the source and

$$\tilde{g}^{(0)}(\boldsymbol{f}, \nu) = (2\pi)^{-2} \int g^{(0)}(\boldsymbol{\rho}', \nu) \exp(-\mathrm{i}\boldsymbol{f}\cdot\boldsymbol{\rho}')\, \mathrm{d}^2\rho' \tag{5.2}$$

is the two-dimensional Fourier transform of the degree of spectral coherence of the light distribution in the source plane. The expression (5.1) was shown to have all the properties attributed to the radiance function in traditional radiometry. It follows from this result and the result established earlier in this section (italicized above) that *when the source is quasi-homogeneous, all the generalized radiance functions $\mathscr{B}_\nu^{(\Omega)}(\boldsymbol{\rho}, \boldsymbol{s})$ of the class that we have considered in this paper have the same asymptotic limit, given by eq. (5.1), as $k \to \infty$; and this common limiting form of all the generalized radiance functions may be identified with the radiance of traditional radiometry, at least at all points in the source plane.*

It should be evident that we have not proven, in the strict mathematical sense that radiometry, even for the restricted class of quasi-homogeneous sources, is the asymptotic limit for large wave numbers of statistical wave theory. In this connection it might be worthwhile to point out that the somewhat analogous statement frequently made that classical mechanics is the limit of quantum mechanics if Planck's constant $h \to 0$ has not been rigorously justified to this day; and that even the restricted class of systems for which this statement may perhaps be true has not been precisely defined. Nevertheless we are of the opinion that the results derived in this note provide a genuine insight into the true meaning of the radiance.

Acknowledgement

It is a pleasure to acknowledge stimulating and helpful discussions with Prof. H.M. Nussenzveig about the subject matter of this note.

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Propagation law for Walther's first generalized radiance function and its short-wavelength limit with quasi-homogeneous sources

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a reprint from Journal of the Optical Society of America A
volume 4, number 7, July 1987

Vol. 4, No. 6/June 1987/J. Opt. Soc. Am. A 1233

Propagation law for Walther's first generalized radiance function and its short-wavelength limit with quasi-homogeneous sources

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Received October 20, 1986; accepted January 20, 1987

An exact law is derived for the propagation in free space of the first generalized radiance function introduced by Walther. A simplified form of this law is obtained for the case when the source is quasi homogeneous. It is also shown that when the source is quasi homogeneous and the wave number is large enough (the wavelength is sufficiently short) the first generalized radiance acquires all the properties of the radiance of traditional radiometry.

1. INTRODUCTION

In two well-known papers[1,2] dealing with the foundation of radiometry, Walther introduced certain generalized radiance functions. These functions have some of but not all the properties that are attributed to the radiance in traditional radiometry.[3] Much subsequent work, aimed at clarifying the connection between radiometry and physical optics, made considerable use of these functions. We will refer to the generalized radiance function introduced in Refs. 1 and 2 as the first and the second generalized radiance functions (g.r.f.'s), respectively.

Approximate transport equations for the propagation of either of these two g.r.f.'s were obtained by Walther,[4] Jannson,[5] Friberg,[6] Pedersen,[7] and Bastiaans.[8] An exact law for the propagation of the second g.r.f. in free space was recently obtained by Foley and Wolf,[9] who also showed that when the source of the optical field is quasi-homogeneous this function acquires, in the short-wavelength limit, all the properties that are postulated for the radiance in traditional radiometry. They also obtained an explicit expression for this *limiting form* of the second g.r.f. in terms of the distribution of the intensity and of the degree of spectral coherence of light across the source.

In the present paper we derive, to begin with, an exact law for propagation of the first g.r.f. in free space. We then consider the form that this propagation law takes when the source is quasi-homogeneous. Finally we consider the asymptotic limit for large wave number (short wavelength) of the first g.r.f. in optical fields produced by a quasi-homogeneous source, and we find that it is identical to the corresponding limiting form obtained for the second g.r.f. in Ref. 9. Under these circumstances the propagation law is found to reduce to the usual radiometric transport equation. These results, together with those derived in Ref. 9, go a long way toward clarifying the foundation of radiometry.

2. PROPAGATION LAW FOR WALTHER'S FIRST GENERALIZED RADIANCE

Let us consider a secondary source occupying a finite domain σ in the plane $z = 0$ and radiating into the half space $z > 0$. We assume that the field fluctuations in the source plane are characterized by a stationary statistical ensemble.

We denote by $W(\mathbf{r}_1, \mathbf{r}_2, \omega)$ the cross-spectral density of the emitted light at any two points P_1 and P_2, specified by position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, in the half-space $z > 0$. Let us choose the two points to be located in some plane z = constant > 0, which we will denote by Π, and let

$$\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2, \qquad \rho = \mathbf{r}_1 - \mathbf{r}_2. \tag{2.1a}$$

Then

$$\mathbf{r}_1 = \mathbf{r} + \rho/2, \qquad \mathbf{r}_2 = \mathbf{r} - \rho/2. \tag{2.1b}$$

(See Fig. 1, where P denotes the point with position vector $\mathbf{r}$.) An expression for the first generalized radiance introduced by Walther,[1] and defined by him at points $\mathbf{r}$ in the source plane $z = 0$, can readily be generalized to apply to field points $\mathbf{r}$ in any transverse plane Π in the half-space $z > 0$. It takes the form

$$\mathcal{B}_\omega(\mathbf{r}, \mathbf{s}) = \left(\frac{k}{2\pi}\right)^2 s_z \int_\Pi W(\mathbf{r} + \rho/2, \mathbf{r} - \rho/2)\exp(ik\mathbf{s}_\perp \cdot \rho)\mathrm{d}^2\rho, \tag{2.2}$$

where

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda} \tag{2.3}$$

is the wave number associated with the frequency ω and wavelength λ, c is the speed of light *in vacuo*, $\mathbf{s} = (s_x, s_y, s_z)$ is a real unit vector, and $\mathbf{s}_\perp = (s_x, s_y, 0)$ is its transverse component (considered a two dimensional vector).

We will now derive an expression for $\mathcal{B}_\omega$ in terms of the value $\mathcal{B}_\omega^{(0)}$ that the generalized radiance [Eq. (2.2)] takes in the source plane $z = 0$. For this purpose we will make use of the following result established not long ago.[10] The cross-spectral density $W(\mathbf{r}_1, \mathbf{r}_2, \omega)$ may be represented in terms of an appropriate ensemble $\{U(\mathbf{r}, \omega)\}$ of monochromatic wave functions [with time dependence $\exp(-i\omega t)$ understood] that propagate from the source plane $z = 0$ into the half-space $z > 0$, in the form

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle U^*(\mathbf{r}_1, \omega)U(\mathbf{r}_2, \omega)\rangle, \tag{2.4}$$

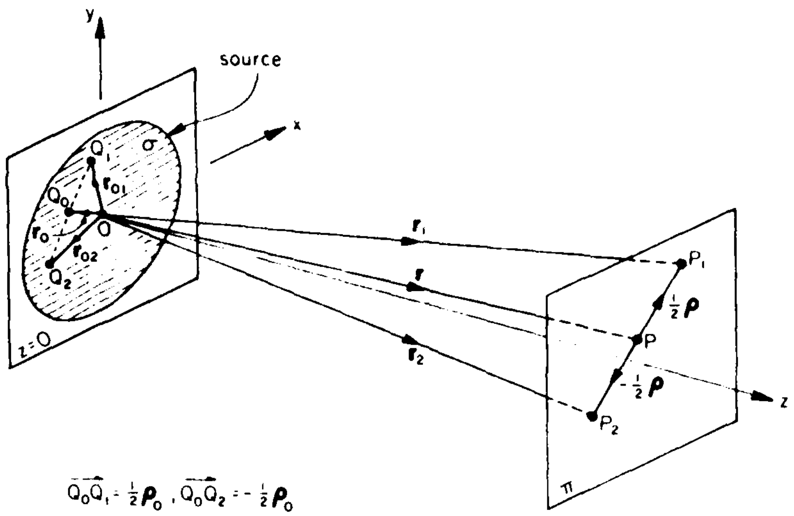

Fig. 1. Illustrating the notation.

Here the angle brackets denote the average taken over this ensemble. Now the value of $U(\mathbf{r}, \omega)$ at any point $\mathbf{r}$ in the half-space $z > 0$ may be expressed in terms of its boundary values $U^{(0)}(\mathbf{r}_0, \omega)$ at point $\mathbf{r}_0$ in the source plane $z = 0$ by the use of Rayleigh's formula[11]

$$U(\mathbf{r}, \omega) = \int_{z=0} G(\mathbf{r} - \mathbf{r}_0, \omega) U^{(0)}(\mathbf{r}_0, \omega) \mathrm{d}^2\mathbf{r}_0, \tag{2.5}$$

where $G(\mathbf{r} - \mathbf{r}_0, \omega)$ is the Green's function

$$G(\mathbf{r} - \mathbf{r}_0, \omega) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\left[\frac{\exp(ik|\mathbf{r} - \mathbf{r}_0|)}{|\mathbf{r} - \mathbf{r}_0|}\right]. \tag{2.6}$$

On substituting from Eq. (2.5) into Eq. (2.4), we readily find that

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \iint_{(z=0)} G^*(\mathbf{R}_1, \omega) G(\mathbf{R}_2, \omega) \times W^{(0)}(\mathbf{r}_{01}, \mathbf{r}_{02}) \mathrm{d}^2\mathbf{r}_{01} \mathrm{d}^2\mathbf{r}_{02}, \tag{2.7}$$

where $\mathbf{r}_{01}$, $\mathbf{r}_{02}$ are the position vectors of two typical points Q_1 and Q_2 in the source plane,

$$\mathbf{R}_j = \mathbf{r}_j - \mathbf{r}_{0j} \qquad (j = 1, 2), \tag{2.8}$$

and

$$W^{(0)}(\mathbf{r}_{01}, \mathbf{r}_{02}, \omega) = \langle U^{(0)*}(\mathbf{r}_{01}, \omega) U^{(0)}(\mathbf{r}_{02}, \omega) \rangle \tag{2.9}$$

is the cross-spectral density of the light in the source plane. The integration on the right-hand side of Eq. (2.7) is taken twice independently over the source plane.

Let us now change the variables in Eq. (2.7) according to the transformation (2.1b) and according to a similar transformation involving the source variables:

$$\mathbf{r}_{01} = \mathbf{r}_0 + \rho_0/2, \qquad \mathbf{r}_{02} = \mathbf{r}_0 - \rho_0/2. \tag{2.10}$$

($\mathbf{r}_0$ represents the point Q_0 in Fig. 1.) The formula (2.7) then takes the form

$$W(\mathbf{r} + \rho/2, \mathbf{r} - \rho/2, \omega) = \iint_{(z=0)} G^*[\mathbf{r} - \mathbf{r}_0 + (\rho - \rho_0)/2, \omega] \times G[\mathbf{r} - \mathbf{r}_0 - (\rho - \rho_0)/2, \omega] \times W^{(0)}(\mathbf{r}_0 + \rho_0/2, \mathbf{r}_0 - \rho_0/2, \omega) \mathrm{d}^2 r_0 \mathrm{d}^2\rho_0, \tag{2.11}$$

where we used the fact that $\mathrm{d}^2r_{01}\mathrm{d}^2r_{02} = \mathrm{d}^2r_0\mathrm{d}^2\rho_0$. Let us now substitute from Eq. (2.11) into the expression (2.2), interchange the order of integrations, and make use of the fact that the generalized radiance $\mathcal{B}_\omega^{(0)}(\mathbf{r}_0, \mathbf{s})$ at points in the source plane is given by [cf. Eq. (2.2)]

$$\mathcal{B}_\omega^{(0)}(\mathbf{r}_0, \mathbf{s}) = \left(\frac{k}{2\pi}\right)^2 s_z \int_{z=0} W^{(0)}(\mathbf{r}_0 + \rho_0/2, \mathbf{r}_0 - \rho_0/2, \omega) \times \exp(ik\mathbf{s}_\perp \cdot \rho_0) \mathrm{d}^2\rho_0, \tag{2.12}$$

where s_z is the z component of the unit vector $\mathbf{s}$. We then obtain the following expression for the generalized radiance $\mathcal{B}_\omega(\mathbf{r}, \mathbf{s})$ at any point $\mathbf{r}$ in the half-space $z > 0$ in terms of its boundary values $\mathcal{B}_\omega^{0}(\mathbf{r}_0, \mathbf{s})$ at points $\mathbf{r}_0$ in the source plane:

$$\mathcal{B}_\omega(\mathbf{r}, \mathbf{s}) = \int_{(z=0)} K(\mathbf{r} - \mathbf{r}_0, \mathbf{s}; \omega) \mathcal{B}_\omega^{(0)}(\mathbf{r}_0, \mathbf{s}) \mathrm{d}^2\mathbf{r}_0. \tag{2.13}$$

The kernel $K(\mathbf{r} - \mathbf{r}_0, \mathbf{s}; \omega)$ in this integral is given by the formula

$$K(\mathbf{r} - \mathbf{r}_0, \mathbf{s}; \omega) = \int_{\Pi} G^*(\mathbf{r} - \mathbf{r}_0 + \rho'/2, \omega) G(\mathbf{r} - \mathbf{r}_0 - \rho'/2, \omega) \times \exp(ik\mathbf{s}_\perp \cdot \rho') \mathrm{d}^2\rho'. \tag{2.14}$$

In deriving this expression we changed the variable of integration from ρ to $\rho' = \rho - \rho_0$.

The formula (2.13) represents an exact law for the propagation of Walther's first radiance function from the source plane $z = 0$ into the half-space $z > 0$.

We will now specialize the general formula (2.13) to fields generated by quasi-homogeneous sources that are of particular interest in connection with the foundation of radiometry.[9]

3. PROPAGATION OF THE GENERALIZED RADIANCE WHEN THE SOURCE IS QUASI-HOMOGENEOUS

When the source is quasi-homogeneous, the cross-spectral density function $W^{(0)}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ has the form[12]

$$W^{(0)}(\mathbf{r}_1, \mathbf{r}_2, \omega) = I^{(0)}\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \omega\right) g^{(0)}(\mathbf{r}_2 - \mathbf{r}_1, \omega), \tag{3.1}$$

where $I^{(0)}(\mathbf{r}, \omega)$ is the intensity distribution and $g^{(0)}(\mathbf{r}', \omega)$ is the degree of spectral coherence of the light in the source plane. $I^{(0)}(\mathbf{r}, \omega)$ is assumed to be a slow function of $\mathbf{r}$, whereas $g^{(0)}(\mathbf{r}', \omega)$ is assumed to be a fast function of $\mathbf{r}'$.

On substituting from Eq. (3.1) into Eq. (2.12), we obtain the following expression for the generalized radiance of a quasi-homogeneous source at any point $\mathbf{r}_0$ in the source plane:

$$\mathcal{B}_\omega^{(0)}(\mathbf{r}_0, \mathbf{s}) = k^2 s_z I^{(0)}(\mathbf{r}_0, \omega) \tilde{g}^{(0)}(k\mathbf{s}_\perp, \omega). \tag{3.2}$$

Here

$$\tilde{g}^{(0)}(\mathbf{f}, \omega) = \frac{1}{(2\pi)^2} \int g^{(0)}(\mathbf{r}', \omega) \exp(-i\mathbf{f} \cdot \mathbf{r}') \mathrm{d}^2\mathbf{r}' \tag{3.3}$$

is the Fourier transform of $g^{(0)}(\mathbf{r}', \omega)$. To determine the generalized radiance of the field at any point $\mathbf{r}$ in the half-space $z > 0$ produced by the quasi-homogeneous source, we substitute from Eq. (3.2) into Eq. (2.13) and find that

$$\mathcal{B}_\omega(\mathbf{r}, \mathbf{s}) = k^2 s_z \tilde{g}^{(0)}(k\mathbf{s}_\perp, \omega) M(\mathbf{r}, \mathbf{s}; \omega), \tag{3.4}$$

where

$$M(\mathbf{r}, \mathbf{s}; \omega) = \int K(\mathbf{r} - \mathbf{r}_0, \mathbf{s}; \omega) I^{(0)}(\mathbf{r}_0, \omega) \mathrm{d}^2\mathbf{r}_0 \tag{3.5}$$

and the kernel K is given by Eq. (2.14).

4. SHORT-WAVELENGTH LIMIT OF THE GENERALIZED RADIANCE OF A FIELD GENERATED BY A QUASI-HOMOGENEOUS SOURCE

Let us now consider the behavior of $\mathcal{B}_\omega(\mathbf{r}, \mathbf{s})$ of a field generated by a quasi-homogeneous source in the short-wavelength limit, more precisely, in the asymptotic limit as the wave number $k = 2\pi/\lambda \rightarrow \infty$. For this purpose we first express the Green's function (2.6) in a more explicit form. On carrying out the differentiation we readily find that

$$G(\mathbf{R}, \omega) = -\frac{1}{2\pi}\left[\left(ik - \frac{1}{R}\right)\frac{z}{R}\right]\frac{e^{ikR}}{R}, \tag{4.1}$$

which, for sufficiently large values of kR, may be approximated by

$$G(\mathbf{R}, \omega) \sim -\frac{ik}{2\pi}\left(\frac{z}{R}\right)\frac{e^{ikR}}{R}. \tag{4.2}$$

Next we substitute from expression (4.2) into the expression (2.14) for the propagation kernel $K(\mathbf{r} - \mathbf{r}_0, \mathbf{s}; \omega)$ and find that when $kR_1 \gg 1$, $kR_2 \gg 1$:

$$K(\mathbf{r} - \mathbf{r}_0, \mathbf{s}; \omega) \approx \left(\frac{kz}{2\pi}\right)^2 \int_{\Pi} \frac{e^{-ikR_1}}{R_1^2} \frac{e^{ikR_2}}{R_2^2} \exp(ik\mathbf{s}_\perp \cdot \boldsymbol{\rho}') \mathrm{d}^2\rho', \tag{4.3}$$

where

$$\mathbf{R}_1 = \mathbf{r} - \mathbf{r}_0 + \boldsymbol{\rho}'/2, \tag{4.4a}$$

$$\mathbf{R}_2 = \mathbf{r} - \mathbf{r}_0 - \boldsymbol{\rho}'/2. \tag{4.4b}$$

Let us next determine the asymptotic approximation as $k \rightarrow \infty$ for the factor $M(\mathbf{r}, \mathbf{s}, \omega)$, defined by Eq. (3.5), which enters the expression (3.4) for the radiance function of a field generated by a quasi-homogeneous source. For this purpose we substitute for $K(\mathbf{r} - \mathbf{r}_0, \mathbf{s}; \omega)$ from expression (4.3) into Eq. (3.5) and introduce the new variables

$$\boldsymbol{\rho}_1 = \mathbf{r}_0 - \boldsymbol{\rho}'/2, \tag{4.5a}$$

$$\boldsymbol{\rho}_2 = \mathbf{r}_0 + \boldsymbol{\rho}'/2. \tag{4.5b}$$

One then readily obtains the following expression for $M(\mathbf{r}, \mathbf{s}, \omega)$:

$$M(\mathbf{r}, \mathbf{s}; \omega) \approx \left(\frac{kz}{2\pi}\right)^2 \int \mathrm{d}^2\rho_2 \frac{\exp(ik|\mathbf{r} - \boldsymbol{\rho}_2|)}{|\mathbf{r} - \boldsymbol{\rho}_2|^2} \exp(ik\mathbf{s}_\perp \cdot \boldsymbol{\rho}_2)$$
$$\times \int \mathrm{d}^2\rho_1 \frac{\exp(-ik|\mathbf{r} - \boldsymbol{\rho}_1|)}{|\mathbf{r} - \boldsymbol{\rho}_1|^2} I^{(0)}\left(\frac{\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2}{2}\right)$$
$$\times \exp(-ik\mathbf{s}_\perp \cdot \boldsymbol{\rho}_1), \tag{4.6}$$

where we have made use of the fact that $\mathrm{d}^2r_0\mathrm{d}^2\rho' = \mathrm{d}^2\rho_1\mathrm{d}^2\rho_2$.

We may express Eq. (4.6) in a more symmetric form by making use of the fact that because the source was assumed to be quasi homogeneous, $I^{(0)}(\boldsymbol{\rho}, \omega)$ will change slowly with $\boldsymbol{\rho}$ for each effective frequency ω that contributes to the source spectrum. Hence we may make the approximation

$$I^{(0)}\left(\frac{\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2}{2}, \omega\right) \approx [I^{(0)}(\boldsymbol{\rho}_1, \omega)]^{1/2}[I^{(0)}(\boldsymbol{\rho}_2, \omega)]^{1/2} \tag{4.7}$$

on the right-hand side of expression (4.6). The resulting expression for $M(\mathbf{r}, \mathbf{s}; \omega)$ may then be written in the form

$$M(\mathbf{r}, \mathbf{s}; \omega) \approx \left(\frac{kz}{2\pi}\right)^2 F(\mathbf{r}, \mathbf{s}; \omega) F^*(\mathbf{r}, \mathbf{s}; \omega), \tag{4.8}$$

where

$$F(\mathbf{r}, \mathbf{s}; \omega) = \int [I^{(0)}(\boldsymbol{\rho}_1, \omega)]^{1/2} \frac{\exp(ik|\mathbf{r} - \boldsymbol{\rho}_1|)}{|\mathbf{r} - \boldsymbol{\rho}_1|^2}$$
$$\times \exp(ik\mathbf{s}_\perp \cdot \boldsymbol{\rho}_1) \mathrm{d}^2\rho_1. \tag{4.9}$$

The asymptotic approximation to the integral on the right-hand side of Eq. (4.9) may be determined by the use of the two-dimensional form of the principle of stationary phase.[13] In carrying out the calculations we ignore the dependence of the source intensity $I^{(0)}(\boldsymbol{\rho}, \omega)$ on ω, for reasons indicated in connection with the approximation (4.7). The result is

$$F(\mathbf{r}, \mathbf{s}; \omega) \sim \frac{2\pi i}{kz}\{I_{(0)}[\boldsymbol{\rho} - (z/s_z)\mathbf{s}_\perp, \omega]\}^{1/2} \exp(ik\mathbf{s} \cdot \mathbf{r}) \quad \text{when } S_0 \in \sigma$$
$$\sim 0 \quad \text{when } S_0 \notin \sigma \tag{4.10}$$

as $k \rightarrow \infty$, where S_0 is the point in the source plane $z = 0$ whose position vector is $\boldsymbol{\rho} - (z/s_z)\mathbf{s}_\perp$ (see Fig. 2). On substituting from expression (4.10) into expression (4.8), we obtain for $M(\mathbf{r}, \mathbf{s}; \omega)$ the asymptotic approximation

$$M(\mathbf{r}, \mathbf{s}; \omega) \sim I^{(0)}[\boldsymbol{\rho} - (z/s_z)\mathbf{s}_\perp, \omega] \quad \text{when } S_0 \in \sigma$$
$$\sim 0 \quad \text{when } S_0 \notin \sigma \tag{4.11}$$

as $k \rightarrow \infty$.

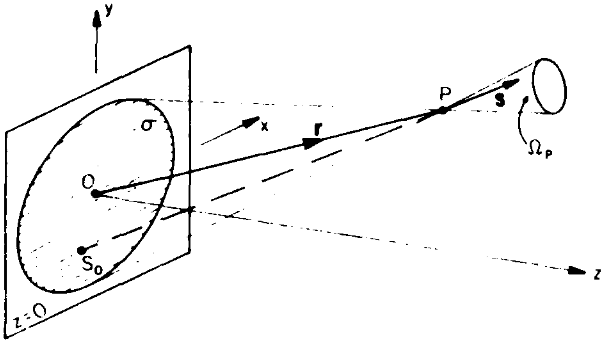

Fig. 2. Illustrating the notation relating to the formula (4.14). S_0 is the point in the source plane at which the line through the field point P in the direction of the unit vector **s** intersects that plane. The vector $\boldsymbol{\rho} - (z/s_z)\mathbf{s}_\perp$, which appears in expressions (4.10), (4.11), and (4.13), is the position vector of the point S_0.

On substituting from expression (4.11) into Eq. (3.4), we see at once that

$$\mathcal{B}_\omega(\mathbf{r}, \mathbf{s}) \sim B_\omega(\mathbf{r}, \mathbf{s}) \quad \text{as } k \to \infty, \tag{4.12}$$

where

$$\begin{aligned} B_\omega(\mathbf{r}, \mathbf{s}) &= k^2 s_z I^{(0)}[\boldsymbol{\rho} - (z/s_z)\mathbf{s}_\perp, \omega] \qquad && \text{when } S_0 \in \sigma \\ &= 0 && \text{when } S_0 \notin \sigma \end{aligned} \tag{4.13}$$

Since $\boldsymbol{\rho} - (z/s_z)\mathbf{s}_\perp$ is the position vector of the point S_0 at which the line through the field point P in the **s** direction intersects the source plane, Eq. (4.13) may be rewritten in the form

$$\begin{aligned} B_\omega(P, \mathbf{s}) &= k^2 s_z I^{(0)}(S_0, \omega)\tilde{g}^{(0)}(k\mathbf{s}_\perp, \omega) \qquad && \text{when } \mathbf{s} \in \Omega_P \\ &= 0 && \text{when } \mathbf{s} \notin \Omega_P, \end{aligned} \tag{4.14}$$

where Ω_P is the solid angle generated by the lines from all the source points to P (see Fig. 2).

The expression (4.14) is identical (except for a slight change in notation) with the expression derived in Ref. 9 for the asymptotic limit as $k \to \infty$ of the second generalized radiance function of Walther, under the assumption, made also in the present paper, that the source of the field is a quasi homogeneous source. It has also been shown in Ref. 9 that the expression (4.14) satisfies all the postulates of traditional radiometry in free space. We may, therefore, conclude by saying that the analysis presented in this paper supports the view that traditional radiometry may be regarded as the asymptotic limit for large wave numbers (short wavelengths) of statistical wave theory of fields produced by quasi homogeneous sources.

ACKNOWLEDGMENTS

This research was supported by the National Science Foundation under grant PHY-8314626 and by the U.S. Air Force Geophysics Laboratory under AFOSR Task 2310G1.

Emil Wolf is also with the Institute of Optics, University of Rochester.

Note added in proof: Since this paper was written another paper dealing with the foundations of radiometry was published [M. Nieto-Vesperinas, "Classical radiometry and radiative transfer theory: a short-wavelength limit of a general mapping of cross-spectral densities in second-order coherence theory," J. Opt. Soc. Am. A **3**, 1354–1359 (1986)]. Unfortunately the main conclusions of that paper are incorrect because the analysis contains several errors. Specifically, the expression (29) of that paper is not the only expression that satisfies Eq. (27). *Moreover, the flux equation (16) does not imply that* $I(\mathbf{r}, \mathbf{s})$ is positive definite, as is stated below Eq. (32).

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Generalized Stokes reciprocity relations for scattering from dielectric objects of arbitrary shape

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a reprint from Journal of the Optical Society of America A
volume 3, number 12, December 1986

Generalized Stokes reciprocity relations for scattering from dielectric objects of arbitrary shape

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Received February 6, 1986; accepted June 18, 1986

The S matrix is first introduced within the framework of the angular spectrum representation of wave fields interacting with linear dielectric bodies of arbitrary shape. By using some universal properties of the S matrix, a number of relations involving certain generalized reflection and transmission coefficients are derived. These relations may be regarded as generalizations of two well-known classic reciprocity relations due to G. G. Stokes. Two reciprocity relations involving the reflection and the transmission coefficients for interaction of a plane electromagnetic wave with a stratified dielectric medium are obtained as special cases.

1. INTRODUCTION

In a classic paper published in 1849, Stokes[1] derived two well-known reciprocity relations involving reflection and transmission of light. More specifically, he considered a plane monochromatic wave incident upon a plane boundary separating two semi-infinite, homogeneous, isotropic dielectric media. Suppose that θ_i and θ_t are the angles of incidence and refraction, respectively, when the wave propagates from the first into the second medium, and that r and t are the corresponding reflection and transmission coefficients. Next consider the situation when the wave is incident at an angle $\theta_i' = \theta_t$ from the second into the first medium, and let ρ and τ be the corresponding reflection and transmission coefficients. The relations derived by Stokes are

$$\tau t + r^2 = 1, \tag{1.1a}$$

$$\rho + r = 0. \tag{1.1b}$$

The relations (1.1) were later generalized to somewhat more complicated situations involving stratified media, and they have played a useful role in optics of thin homogeneous films.[2] More recently, relations of this kind have become of importance in some investigations concerning the cancellation of distortions by the technique of phase conjugation.[3,4] All these situations have one feature in common. They involve a homogeneous or a succession of homogeneous dielectrics with mutually parallel planar boundaries, and, consequently, when a plane wave is incident upon such a configuration only one reflected and one transmitted wave is generated. It seems natural to inquire whether one can *generalize the Stokes relations further*, so that they apply to situations such as rough-surface scattering and scattering from an inhomogeneous plane-parallel dielectric slab or to phase conjugation of waves that are scattered from a dielectric body of arbitrary shape. In the present paper we obtain such a generalization within the framework of the scalar wave theory. Our derivation utilizes in a basic way the concepts of the angular spectrum representation of wave fields and of the S matrix. The combined use of these two concepts has already proved rather useful in treatments of other problems, which yielded interesting results relating to *the theory of antennas*[5] *and to distortion correction by phase conjugation.*[6] The generalization of the Stokes relations presented in this paper does not, however, appear to have been obtained previously.

2. SOME GENERAL RELATIONS INVOLVING THE ANGULAR SPECTRUM REPRESENTATION OF WAVE FIELDS AND THE S MATRIX

Consider a monochromatic field, not necessarily a planar one, incident upon a dielectric scatterer. We denote by $U^{(i)}(\mathbf{r})\exp(-i\omega t)$ and $U^{(s)}(\mathbf{r})\exp(-i\omega t)$ the incident and the scattered fields, respectively, with $\mathbf{r}$ denoting the position vector of a typical point in space, t the time, and ω the frequency. The total field $U(\mathbf{r})\exp(-i\omega t)$ is, of course, the sum of these two fields.

Let us choose a Cartesian-coordinate system of axis so that the scatterer is situated within the strip $0 \le z \le L$, and let $\mathcal{R}^-$ and $\mathcal{R}^+$ be the two half-spaces on the two sides of the scatterer (see Fig. 1). It is well known that under very general conditions the total field in each of the two half-spaces may be represented in the form of an angular spectrum of plane waves, both homogeneous and evanescent ones.[7] The amplitudes of the evanescent waves decay exponentially with increasing distance from the scatterer. Because we will be interested only in the field far away from the scatterer, we will omit the contributions of the evanescent waves. The angular spectrum representation of the time-independent part of the total field then takes the following form in the two half spaces:

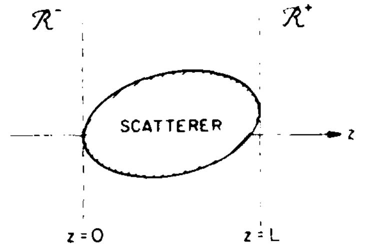

Fig. 1. *Illustrating the notation.*

In $\mathcal{R}^-$:

$$U(\mathbf{r}) = -\frac{ik}{2\pi}\int_{\sigma^{(+)}} C^{(-)}(\mathbf{n})e^{ik\mathbf{n}\cdot\mathbf{r}}d\Omega + \frac{ik}{2\pi}\int_{\sigma^{(-)}} D^{(-)}(\mathbf{n})e^{ik\mathbf{n}\cdot\mathbf{r}}d\Omega, \tag{2.1a}$$

In $\mathcal{R}^+$:

$$U(\mathbf{r}) = -\frac{ik}{2\pi}\int_{\sigma^{(-)}} C^{(+)}(\mathbf{n})e^{ik\mathbf{n}\cdot\mathbf{r}}d\Omega + \frac{ik}{2\pi}\int_{\sigma^{(+)}} D^{(+)}(\mathbf{n})e^{ik\mathbf{n}\cdot\mathbf{r}}d\Omega. \tag{2.1b}$$

In these formulas $\mathbf{n} = (n_x, n_y, n_z)$ are real unit vectors, $k = \omega/c$ is the wave number associated with the frequency ω (c being the speed of light *in vacuo*), $d\Omega$ is the element of solid angle generated by the unit vector $\mathbf{n}$, and $\sigma^{(+)}$ and $\sigma^{(-)}$ are unit hemispheres in $\mathbf{n}$ space defined as

$$\sigma^{(+)}:\quad n^2 = 1, \qquad n_z \geq 0, \tag{2.2a}$$

$$\sigma^{(-)}:\quad n^2 = 1, \qquad n_z < 0. \tag{2.2b}$$

The constants multiplying the integrals in Eqs. (2.1) have been chosen so as to simplify subsequent formulas.

In the representation (2.1) the factors $C^{(\pm)}$ and $D^{(\pm)}$ have the physical significance of (generally complex) amplitudes of homogeneous plane waves that propagate in different directions either toward the scatterer [waves with amplitudes $C^{(+)}$ and $C^{(-)}$] or away from it [waves with amplitudes $D^{(+)}$ *and* $D^{(-)}$]. However, they also have another physical significance, which becomes evident when one examines the behavior of the total field far away from the scatterer. One then finds, for example by the use of the principle of stationary phase,[8] that as the distance r of the field point from the fixed origin O in the source region increases along any fixed direction specified by any real unit vector $\mathbf{u} = (u_x, u_y, u_z)$,

$$U(r\mathbf{u}) \sim C^{(\pm)}(-\mathbf{u})\frac{e^{-ikr}}{r} + D^{(\pm)}(\mathbf{u})\frac{e^{ikr}}{r} \qquad \text{as } kr \to \infty, \tag{2.3}$$

where the upper or the lower signs are taken on the right-hand side according to whether the field point $\mathbf{r} = r\mathbf{u}$ is located in the half-space $\mathcal{R}^+$ or $\mathcal{R}^-$, i.e., according to whether $u_z > 0$ or $u_z < 0$.

The formula (2.3) expresses the far field in each of the two half-spaces on either side of the scatterer as a sum of a converging and a diverging spherical wave, with complex amplitudes $C^{(\pm)}$ and $D^{(\pm)}$ (see Fig. 2). This result implies that the integrals in Eqs. (2.1) that contain the (generally complex) spectral amplitudes $C^{(\pm)}$ represent a field that is incoming at infinity, whereas the integrals containing the spectral amplitudes $D^{(\pm)}$ represent a field that is outgoing at infinity.[9]

We will assume that the scatterer is *linear*, i.e., that the outgoing field depends linearly on the incoming field. Consequently the C amplitudes and the D amplitudes will be coupled by a relation of the form[10]

$$\mathbf{D}(\mathbf{n}) = -\int \mathbf{S}(\mathbf{n}, \mathbf{n}')\mathbf{C}(\mathbf{n}')d\Omega', \tag{2.4}$$

where $\mathbf{C}$ and $\mathbf{D}$ are the column vectors

$$\mathbf{C}(\mathbf{n}) = \begin{bmatrix} C^{(-)}(\mathbf{n}) \\ C^{(+)}(\mathbf{n}) \end{bmatrix}, \qquad \mathbf{D}(\mathbf{n}) = \begin{bmatrix} D^{(+)}(\mathbf{n}) \\ D^{(-)}(\mathbf{n}) \end{bmatrix} \tag{2.5}$$

and $\mathbf{S}$ is, *for each pair of arguments* $\mathbf{n}$ and $\mathbf{n}'$, a 2×2 matrix. Written out more explicitly, Eq. (2.4) gives

$$D^{(+)}(\mathbf{n}) = -\int_{\sigma^{(+)}} S^{(+-)}(\mathbf{n}, \mathbf{n}')C^{(-)}(\mathbf{n}')d\Omega' - \int_{\sigma^{(-)}} S^{(++)}(\mathbf{n}, \mathbf{n}')C^{(+)}(\mathbf{n}')d\Omega', \tag{2.6a}$$

$$D^{(-)}(\mathbf{n}) = -\int_{\sigma^{(+)}} S^{(--)}(\mathbf{n}, \mathbf{n}')C^{(-)}(\mathbf{n}')d\Omega' - \int_{\sigma^{(-)}} S^{(-+)}(\mathbf{n}, \mathbf{n}')C^{(+)}(\mathbf{n}')d\Omega', \tag{2.6b}$$

with

$$\mathbf{S}(\mathbf{n}, \mathbf{n}') = \begin{bmatrix} S^{(+-)}(\mathbf{n}, \mathbf{n}') & S^{(++)}(\mathbf{n}, \mathbf{n}') \\ S^{(--)}(\mathbf{n}, \mathbf{n}') & S^{(-+)}(\mathbf{n}, \mathbf{n}') \end{bmatrix}. \tag{2.7}$$

From the significance of the quantities $C^{(\pm)}$ and $D^{(\pm)}$ as complex amplitudes of waves that propagate either toward or away from the scatterer, and recalling the definitions (2.2) of the domains of integration in Eqs. (2.6), it is clear that the four elements of the 2×2 matrix (2.2) are defined only for the following ranges of the z components of the unit vectors $\mathbf{n}$ and $\mathbf{n}'$:

$$S^{(+-)}(\mathbf{n}, \mathbf{n}'):\qquad n_z > 0, \quad n_z' > 0, \tag{2.8a}$$

$$S^{(++)}(\mathbf{n}, \mathbf{n}'):\qquad n_z > 0, \quad n_z' < 0, \tag{2.8b}$$

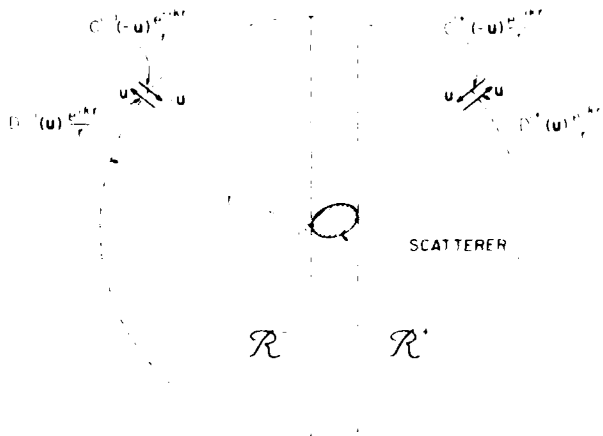

Fig. 2. The far fields in the half spaces $\mathcal{R}^+$ and $\mathcal{R}^-$ on either side of the scatterer.

$$S^{(-+)}(\mathbf{n}, \mathbf{n}'): \quad n_z < 0, \quad n_z' > 0, \tag{2.8c}$$

$$S^{(--)}(\mathbf{n}, \mathbf{n}'): \quad n_z < 0, \quad n_z' < 0. \tag{2.8d}$$

It will be useful at this point to make a contact with the usual S matrix of potential scattering. In the theory of potential scattering the expression for the far field, which corresponds to expression (2.3), would be written in the form

$$U(r\mathbf{u}) \sim F_1(\mathbf{u})\frac{e^{ikr}}{r} + F_2(\mathbf{u})\frac{e^{-ikr}}{r}, \quad \text{as } kr \to \infty, \tag{2.9}$$

and one expresses the relation between the complex amplitudes of the outgoing and incoming waves in the following form, which corresponds to Eq. (2.4):

$$F_1(\mathbf{n}) = -\int_{\sigma} \mathcal{S}(\mathbf{n}, \mathbf{n}')F_2(-\mathbf{n}')\mathrm{d}\Omega', \tag{2.10}$$

where the integration extends over the whole unit sphere generated by the unit vector $\mathbf{n}'$ [cf. Ref. 11, Eqs. (20) and (21)].

It is clear that $\mathcal{S}(\mathbf{n}, \mathbf{n}')$ and $\boldsymbol{S}(\mathbf{n}, \mathbf{n}')$ are essentially the same quantities, both being continuous matrices whose elements are labeled by pairs of real unit vectors $\mathbf{n}$ and $\mathbf{n}'$. However, in contrast with $\mathcal{S}(\mathbf{n}, \mathbf{n}')$, each element of $\boldsymbol{S}(\mathbf{n}, \mathbf{n}')$ is itself a matrix, arising from the partition of $\mathcal{S}(\mathbf{n}, \mathbf{n}')$ into four separate contributions [Eqs. (2.7)]. Such partition is advantageous when the field in each of the two half-spaces R^- and R^+ is represented in the form of an angular spectrum of plane waves.

For later purposes we recall some general properties of the usual scattering matrix $\mathcal{S}(\mathbf{n}, \mathbf{n}')$. It is well known that when the scatterer is dielectric (i.e., lossless), $\mathcal{S}$ is unitary, i.e., it obeys the relations [cf. Ref. 11, Eqs. (24) and (29)]

$$\int_{\sigma} \mathcal{S}^*(\mathbf{n}, \mathbf{n}')\mathcal{S}(\mathbf{n}, \mathbf{n}'')\mathrm{d}\Omega = \Delta(\mathbf{n}' - \mathbf{n}''), \tag{2.11a}$$

$$\int_{\sigma} \mathcal{S}(\mathbf{n}', \mathbf{n})\mathcal{S}^*(\mathbf{n}'', \mathbf{n})\mathrm{d}\Omega = \Delta(\mathbf{n}' - \mathbf{n}''), \tag{2.11b}$$

where the asterisk denotes the complex conjugate and the integrations extend over the unit sphere generated by the unit vector $\mathbf{n}$. Further, $\Delta(\mathbf{n}' - \mathbf{n}'')$ is the "spherical" Dirac delta function, defined by the formula

$$\Delta(\mathbf{n}' - \mathbf{n}'') = \frac{\delta(\theta' - \theta'')\delta(\varphi' - \varphi'')}{|\sin\theta'|}, \tag{2.12}$$

where (θ', φ') and (θ'', φ'') are the spherical polar coordinates of the unit vectors $\mathbf{n}'$ and $\mathbf{n}''$, respectively, and δ is the usual one-dimensional Dirac delta function.

The $\mathcal{S}$ matrix also obeys the reciprocity relation [Ref. 11, Eq. (28)]

$$\mathcal{S}(-\mathbf{n}', -\mathbf{n}) = \mathcal{S}(\mathbf{n}, \mathbf{n}'). \tag{2.13}$$

We show in Appendix B that when the incident field is a plane wave,

$$U^{(i)}(\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}} \quad (\mathbf{n}_0^2 = 1), \tag{2.14}$$

the factors F_1 and F_2 in the asymptotic approximation (2.9) of the total field (incident + scattered) are given by

$$F_1(\mathbf{n}) = \frac{2\pi}{ik}\mathcal{S}(\mathbf{n}, \mathbf{n}_0), \tag{2.15a}$$

$$F_2(\mathbf{n}) = -\frac{2\pi}{ik}\Delta(\mathbf{n} + \mathbf{n}_0). \tag{2.15b}$$

3. THE GENERALIZED TRANSMISSION AND REFLECTION COEFFICIENTS

We will now show that the four elements of the partitioned S matrix that we introduced through Eqs. (2.6) have a simple interpretation.

Suppose that a monochromatic plane wave of unit amplitude and direction of propagation specified by a real unit vector $\mathbf{n}_0$, i.e.,

$$U^{(i)}(\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}} \tag{3.1}$$

[with time periodic factor $\exp(-i\omega t)$ not shown], is incident upon the scatterer. It then follows from Eqs. (2.15), (2.9), and (2.3) that

$$C^{(\pm)}(\mathbf{n}) = -\frac{2\pi}{ik}\Delta(\mathbf{n} - \mathbf{n}_0), \tag{3.2a}$$

$$D^{(\pm)}(\mathbf{n}) = \frac{2\pi}{ik}\mathcal{S}(\mathbf{n}, \mathbf{n}_0), \tag{3.2b}$$

the upper or lower signs being taken on the left-hand sides of these formulas according to whether $n_z \gtrless 0$. Now the second integral on the right-hand sides of Eqs. (2.1) represents the outgoing field, $U^{(\mathrm{out})}(\mathbf{r}; \mathbf{n}_0)$ say, i.e., the field that behaves as a diverging spherical wave at infinity. Hence it follows, on making use of Eq. (3.2b), that when the plane wave given by Eq. (3.1) is incident upon the scattering object

$$U^{(\mathrm{out})}(\mathbf{r}; \mathbf{n}_0) = \int_{\sigma^{(-)}} \mathcal{S}(\mathbf{n}, \mathbf{n}_0)e^{ik\mathbf{n}\cdot\mathbf{r}}\mathrm{d}\Omega \quad \text{when } \mathbf{r} \in R^-, \tag{3.3a}$$

$$= \int_{\sigma^{(+)}} \mathcal{S}(\mathbf{n}, \mathbf{n}_0)e^{ik\mathbf{n}\cdot\mathbf{r}}\mathrm{d}\Omega \quad \text{when } \mathbf{r} \in R^+, \tag{3.3b}$$

where $\sigma^{(-)}$ and $\sigma^{(+)}$ are the hemispheres defined by Eqs. (2.2).

The formulas (3.3) represent the outgoing field in each of the two half-spaces R^- and R^+ in the form of an angular spectrum of plane waves, with (generally complex) amplitudes $\mathcal{S}(\mathbf{n}, \mathbf{n}_0)$ that propagate away from the scatterer in directions specified by unit vectors $\mathbf{n}$. When the z component, n_{0z}, of the unit propagation vector $\mathbf{n}_0$ of the incident wave is positive $\mathcal{S}(\mathbf{n}, \mathbf{n}_0)$ clearly has the physical significance of a *generalized transmission coefficient*, $t(\mathbf{n}, \mathbf{n}_0)$ say, when $n_z \geq 0$ and of a *generalized reflection coefficient*, $r(\mathbf{n}, \mathbf{n}_0)$ say, when $n_z < 0$, *for incidence from the half-space* R^- (see Fig. 3). Recalling expressions (2.8a) and (2.8b), we see that these coefficients are precisely two of the elements of the partitioned S matrix (2.7), viz.,

$$t(\mathbf{n}, \mathbf{n}_0) = S^{(++)}(\mathbf{n}, \mathbf{n}_0), \quad n_z > 0, \quad n_{0z} > 0, \tag{3.4a}$$

$$r(\mathbf{n}, \mathbf{n}_0) = S^{(-+)}(\mathbf{n}, \mathbf{n}_0), \quad n_z < 0, \quad n_{0z} > 0. \tag{3.4b}$$

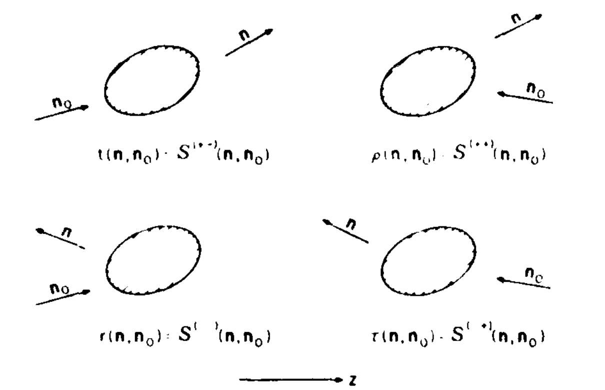

Fig. 3. Illustrating the significance of the elements of the partitioned S matrix as generalized transmission and reflection coefficients.

Similarly the quantities

$$\tau(\mathbf{n}, \mathbf{n}_0) = S^{(-+)}(\mathbf{n}, \mathbf{n}_0), \qquad n_z < 0, \quad n_{0z} < 0, \tag{3.4c}$$

$$\rho(\mathbf{n}, \mathbf{n}_0) = S^{(++)}(\mathbf{n}, \mathbf{n}_0), \qquad n_z > 0, \quad n_{0z} < 0 \tag{3.4d}$$

have the physical significance of a *generalized transmission coefficient* and a *generalized reflection coefficient*, respectively, *for incidence from the half space* R^+ (see Fig. 3).

It follows that in terms of these generalized transmission and reflection coefficients, the partitioned S matrix (2.7) may be expressed in the form

$$\boldsymbol{S}(\mathbf{n}, \mathbf{n}') = \begin{bmatrix} t(\mathbf{n}, \mathbf{n}') & \rho(\mathbf{n}, \mathbf{n}') \\ r(\mathbf{n}, \mathbf{n}') & \tau(\mathbf{n}, \mathbf{n}') \end{bmatrix}. \tag{3.5}$$

It is to be noted that in view of the relation (2.13), the generalized transmission and reflection coefficients obey the *reciprocity relations*

$$t(-\mathbf{n}', -\mathbf{n}) = \tau(\mathbf{n}, \mathbf{n}'), \tag{3.6a}$$

$$\tau(-\mathbf{n}', -\mathbf{n}) = t(\mathbf{n}, \mathbf{n}'), \tag{3.6b}$$

$$\rho(-\mathbf{n}', -\mathbf{n}) = \rho(\mathbf{n}, \mathbf{n}'), \tag{3.6c}$$

$$r(-\mathbf{n}', -\mathbf{n}) = r(\mathbf{n}, \mathbf{n}'). \tag{3.6d}$$

It seems worthwhile to point out that our definition of the generalized transmission and reflection coefficients depends on the choice of the z axis. It is possible for a transmission coefficient defined with respect to a particular z direction to become a reflection coefficient, and vice versa, when the z direction is chosen differently. However, in many situations of practical interest a particular direction is distinguished from all other directions, and it is then natural to choose the z axis along this special direction. Examples include stratified media, inhomogeneous plane-parallel plates, and rough planar surfaces. Scattering from bodies of arbitrary shape in the presence of planar phase-conjugate mirrors also belongs in this category.

4. GENERALIZED STOKES RELATIONS

With the interpretation of the elements of the partitioned S matrix that we just discussed, we are now in a position to formulate certain generalized Stokes relations. For this purpose we first separate in the integral that expresses the unitary condition (2.11a) of $\mathscr{S}(\mathbf{n}, \mathbf{n}')$ the contributions from the hemispheres $\sigma^{(-)}$ and $\sigma^{(+)}$, defined by Eqs. (2.2):

$$\int_{\sigma^{(-)}} \mathscr{S}^*(\mathbf{n}, \mathbf{n}')\mathscr{S}(\mathbf{n}, \mathbf{n}'')d\Omega + \int_{\sigma^{(+)}} \mathscr{S}^*(\mathbf{n}, \mathbf{n}')\mathscr{S}(\mathbf{n}, \mathbf{n}'')d\Omega = \Delta(\mathbf{n}' - \mathbf{n}''). \tag{4.1}$$

If we chose the unit vectors $\mathbf{n}'$ and $\mathbf{n}''$ to have positive z components, i.e., $n_z' > 0$ and $n_z'' > 0$, and recall the physical significance of $\mathscr{S}(\mathbf{n}, \mathbf{n}')$ and $\mathscr{S}(\mathbf{n}, \mathbf{n}'')$ discussed in Section 3, we obtain at once from Eq. (4.1) the following relation:

$$\int_{\sigma^{(-)}} r^*(\mathbf{n}, \mathbf{n}')r(\mathbf{n}, \mathbf{n}'')d\Omega + \int_{\sigma^{(+)}} t^*(\mathbf{n}, \mathbf{n}')t(\mathbf{n}, \mathbf{n}'')d\Omega = \Delta(\mathbf{n}' - \mathbf{n}''). \tag{4.2a}$$

Next let us choose the unit vectors $\mathbf{n}'$ and $\mathbf{n}''$ with $n_z' < 0$ and $n_z'' > 0$. The formula (4.1) then gives, if we also use Eqs. (3.4),

$$\int_{\sigma^{(-)}} \tau^*(\mathbf{n}, \mathbf{n}')r(\mathbf{n}, \mathbf{n}'')d\Omega + \int_{\sigma^{(+)}} \rho^*(\mathbf{n}, \mathbf{n}')t(\mathbf{n}, \mathbf{n}'')d\Omega = 0. \tag{4.2b}$$

In a similar way we obtain with the choice $n_z' > 0$, $n_z'' < 0$

$$\int_{\sigma^{(-)}} r^*(\mathbf{n}, \mathbf{n}')\tau(\mathbf{n}, \mathbf{n}'')d\Omega + \int_{\sigma^{(+)}} t^*(\mathbf{n}, \mathbf{n}')\rho(\mathbf{n}, \mathbf{n}'')d\Omega = 0, \tag{4.2c}$$

and, with the choice $n_z' < 0$, $n_z'' < 0$,

$$\int_{\sigma^{(-)}} \tau^*(\mathbf{n}, \mathbf{n}')\tau(\mathbf{n}, \mathbf{n}'')d\Omega + \int_{\sigma^{(+)}} \rho^*(\mathbf{n}, \mathbf{n}')\rho(\mathbf{n}, \mathbf{n}'')d\Omega = \Delta(\mathbf{n}' - \mathbf{n}''). \tag{4.2d}$$

One can readily verify that the four relations (4.2) are equivalent to the following matrix equation, which expresses the unitarity condition (2.11a) in terms of our partitioned S matrix in a familiar form:

$$\int_{\sigma} \boldsymbol{S}^\dagger(\mathbf{n}, \mathbf{n}')\boldsymbol{S}(\mathbf{n}, \mathbf{n}'')d\Omega = \boldsymbol{I}\Delta(\mathbf{n}' - \mathbf{n}''). \tag{4.3}$$

Here $\boldsymbol{S}^\dagger$ is the Hermitian adjoint of $\boldsymbol{S}$ and $\boldsymbol{I}$ is the identity matrix.

In a similar manner that led to the relations (4.2) one can derive from the second unitarity condition (2.11b) of $\mathscr{S}(\mathbf{n}, \mathbf{n}')$ the following four relations:

$$\int_{\sigma^{(-)}} \rho(\mathbf{n}', \mathbf{n})\rho^*(\mathbf{n}'', \mathbf{n})d\Omega + \int_{\sigma^{(+)}} t(\mathbf{n}', \mathbf{n})t^*(\mathbf{n}'', \mathbf{n})d\Omega = \Delta(\mathbf{n}' - \mathbf{n}''), \tag{4.4a}$$

$$\int_{\sigma^{(-)}} r(\mathbf{n}', \mathbf{n})\rho^*(\mathbf{n}'', \mathbf{n})d\Omega + \int_{\sigma^{(+)}} \tau(\mathbf{n}', \mathbf{n})t^*(\mathbf{n}'', \mathbf{n})d\Omega = 0, \tag{4.4b}$$

$$\int_{\sigma^{(-)}} \rho(\mathbf{n}', \mathbf{n})\tau^*(\mathbf{n}'', \mathbf{n})d\Omega + \int_{\sigma^{(+)}} t(\mathbf{n}', \mathbf{n})r^*(\mathbf{n}'', \mathbf{n})d\Omega = 0. \tag{4.4c}$$

$$\int_{\sigma^{(-)}} t(\mathbf{n}', \mathbf{n})t^*(\mathbf{n}'', \mathbf{n})d\Omega + \int_{\sigma^{(+)}} r(\mathbf{n}', \mathbf{n})r^*(\mathbf{n}'', \mathbf{n})d\Omega = \Delta(\mathbf{n}' - \mathbf{n}''). \quad (4.4d)$$

The relations (4.4) may readily be shown to be equivalent to the following matrix equation, which expresses the second unitary condition (2.1b) in terms of our partitioned S matrix:

$$\int_{\sigma} \boldsymbol{S}(\mathbf{n}', \mathbf{n})\boldsymbol{S}^{\dagger}(\mathbf{n}'', \mathbf{n})d\Omega = \boldsymbol{I}\Delta(\mathbf{n}', \mathbf{n}''). \quad (4.5)$$

The formulas (4.2) and (4.4) may be regarded as generalizations of the Stokes reciprocity relations (1.1). We verify in Section 5 that they reduce to Eqs. (1.1) in the special case considered by Stokes.

Of the eight relations (4.2) and (4.4) only two are actually independent of each other. To see this let us first apply the reciprocity relations (3.6a) and (3.6c) to Eq. (4.4a) and take the complex conjugate of the resulting equation. This gives

$$\int_{\sigma^{(-)}} \rho^*(-\mathbf{n}, -\mathbf{n}')\rho(-\mathbf{n}, -\mathbf{n}'')d\Omega + \int_{\sigma^{(+)}} \tau^*(-\mathbf{n}, -\mathbf{n}')\tau(-\mathbf{n}, -\mathbf{n}'')d\Omega = \Delta(\mathbf{n}' - \mathbf{n}''). \quad (4.6)$$

If we now change the variables by letting $\mathbf{n} \to -\mathbf{n}$, $\mathbf{n}' \to -\mathbf{n}'$, and $\mathbf{n}'' \to -\mathbf{n}''$, the relation (4.6) becomes

$$\int_{\sigma^{(+)}} \rho^*(\mathbf{n}, \mathbf{n}')\rho(\mathbf{n}, \mathbf{n}'')d\Omega + \int_{\sigma^{(-)}} \tau^*(\mathbf{n}, \mathbf{n}')\tau(\mathbf{n}, \mathbf{n}'')d\Omega = \Delta(\mathbf{n}' - \mathbf{n}''), \quad (4.7)$$

which is the relation (4.2d). In a strictly similar manner one can show, with the help of the reciprocity relations (3.6), that Eqs. (4.4b), (4.4c), and (4.4d) are equivalent to Eqs. (4.2c), (4.2b), and (4.2a), respectively. Hence if we take the reciprocity relations (3.6) into account, the set of the four equations (4.2) contains the same information as the set (4.4). We may, therfore, confine our attention from now on to the set (4.2) only.

Since the two half-spaces $\mathscr{R}^-$ and $\mathscr{R}^+$ play the same role in the present theory, it is clear that from any of the generalized Stokes relations that we just derived one will obtain a valid relation through the simultaneous transformations

$$\sigma^{(+)} \leftrightarrow \sigma^{(-)}, \quad (4.8a)$$

$$t \leftrightarrow \tau, \quad (4.8b)$$

$$\rho \leftrightarrow r. \quad (4.8c)$$

Two formulas that transform into each other in this way may be said to be *dual* of each other. Clearly Eqs. (4.2a) and (4.2d) form a dual pair, and so do Eqs. (4.2b) and (4.2c). *Hence there are essentially only two independent relations* of the type that we are considering, which we may take to be Eqs. (4.2a) and (4.2b). The other six relations may be obtained from them by the use of reciprocity and duality.

5. AN EXAMPLE: STOKES RELATIONS FOR STRATIFIED DIELECTRIC MEDIA SURROUNDED BY FREE SPACE

We will illustrate the use of the general relations (4.2a) and (4.2b) by applying them to the interaction of a plane monochromatic electromagnetic wave with a stratified dielectric medium.

Consider a stratified dielectric medium that occupies the strip $0 \leq z \leq L$, and let $N = N(z)$ be the (real) refractive-index function of the medium. We assume that the stratified medium is surrounded by free space; hence $N(z) = 1$ when $z < 0$ and when $z > L$. We assume further that the incident electromagnetic wave is linearly polarized, with its electric field either in the plane of incidence (TM wave) or *perpendicular to it (TE wave). As is well known [Ref. 12, Sec. 1.6.1]* the state of polarization of either of these two waves (modes) does not change on interaction with the stratified medium; and an incident wave of any state of polarization may be expressed as a linear combination of these two modes, which, moreover, are independent of each other when they interact with the stratified medium.

A. Consequences of Eq. (4.2a)

Suppose first that the wave is incident upon the stratified medium from the half-space $z < 0$, in a direction specified by *a unit vector* $\mathbf{n}'(n_z' > 0)$, *and let* $\mathbf{n}_r(n')$ *and* $\mathbf{n}_t(\mathbf{n}')$ *be the unit* vectors in the direction of propagation of the reflected and the transmitted waves, respectively [Fig. 4(a)]. The functional dependences of $\mathbf{n}_r$ and of $\mathbf{n}_t$ on $\mathbf{n}'$ are given by the laws of reflection and refraction, respectively, for stratified media (Ref. 12, Secs. 1.6.1 and 1.6.3). Since there is only one reflected and one transmitted plane wave, the generalized reflection and transmission coefficients will evidently be of the form

$$r(\mathbf{n}, \mathbf{n}') = \bar{r}(\mathbf{n}')\Delta[\mathbf{n} - \mathbf{n}_r(\mathbf{n}')] \quad (5.1a)$$

and

$$t(\mathbf{n}, \mathbf{n}') = \bar{t}(\mathbf{n}')\Delta[\mathbf{n} - \mathbf{n}_t(\mathbf{n}')], \quad (5.1b)$$

where $\bar{r}$ and $\bar{t}$ are the usual reflection and transmission coef-

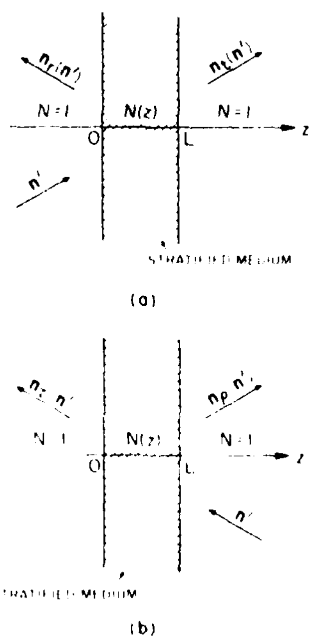

Fig. 4 Illustrating the notation relating to the derivation of the Stokes relations for stratified dielectric media

[illegible]cients, respectively, for incidence for the half-space $z < 0$. Let us consider the first term on the left-hand side of the generalized Stokes relation (4.2a). If we use Eq. (5.1a) we have

$$\int r^*(\mathbf{n}, \mathbf{n}')r(\mathbf{n}', \mathbf{n}'')d\Omega'$$

$$= \int r^*(\mathbf{n}')r(\mathbf{n}'')\Delta[\mathbf{n} - \mathbf{n}_r(\mathbf{n}')]\Delta[\mathbf{n} - \mathbf{n}_r(\mathbf{n}'')]d\Omega. \quad (5.2)$$

Let (θ, φ), (θ', φ'), and (θ'', φ'') be the spherical polar angles of the unit vectors $\mathbf{n}$, $\mathbf{n}'$, and $\mathbf{n}''$, with the polar axis being taken along the positive z direction. Then we have, according to the law of reflection,

$$n_r(\mathbf{n}') = (\pi - \theta', \varphi'), \qquad n_r(\mathbf{n}'') = (\pi - \theta'', \varphi''), \quad (5.3)$$

and Eq. (5.2) becomes

$$\int r^*(\mathbf{n}, \mathbf{n}')r(\mathbf{n}', \mathbf{n}'')d\Omega'$$

$$= r^*(\mathbf{n}')r(\mathbf{n}'') \int \Delta[\mathbf{n} - \mathbf{n}_r(\mathbf{n}')]\Delta[\mathbf{n} - \mathbf{n}_r(\mathbf{n}'')]d\Omega. \quad (5.4)$$

The integral on the right-hand side may be evaluated at once [illegible] use of Eq. (C3) of Appendix C and the fact that $n_z < 0$. This gives

$$\int \Delta[\mathbf{n} - \mathbf{n}_r(\mathbf{n}')]\Delta[\mathbf{n} - \mathbf{n}_r(\mathbf{n}'')]d\Omega = \Delta[\mathbf{n}_r(\mathbf{n}') - \mathbf{n}_r(\mathbf{n}'')]. \quad (5.5)$$

If we recall the definition [illegible] of the "spherical" delta [illegible] and make use of Eq. (5.3) and the fact that $n_z < 0$, we obtain from Eq. (5.5) the relation

$$\int \Delta[\mathbf{n} - \mathbf{n}_r(\mathbf{n}')]\Delta[\mathbf{n} - \mathbf{n}_r(\mathbf{n}'')]d\Omega$$

$$= \frac{\delta[\pi - \theta' - (\pi - \theta'')]\delta(\varphi - \varphi')}{\sin\theta'}$$

$$= \Delta(\mathbf{n}' - \mathbf{n}''). \quad (5.6)$$

On substituting from Eq. (5.6) into Eq. (5.4) we find that

$$\int r^*(\mathbf{n}, \mathbf{n}')r(\mathbf{n}, \mathbf{n}'')d\Omega = r^*(\mathbf{n}')r(\mathbf{n}'')\Delta(\mathbf{n}' - \mathbf{n}''). \quad (5.7)$$

In a strictly similar manner we may evaluate the second [illegible] on the left-hand side of Eq. (4.2a). We then find

$$\int t^*(\mathbf{n}, \mathbf{n}')t(\mathbf{n}, \mathbf{n}'')d\Omega = t^*(\mathbf{n}')t(\mathbf{n}'')\Delta(\mathbf{n}' - \mathbf{n}''). \quad (5.8)$$

[illegible] substituting from Eqs. (5.7) and (5.8) into the [illegible] Stokes relation (4.2a) and integrating both sides [illegible] unit sphere generated by the vector $\mathbf{n}'' = (\theta'', \varphi'')$, we [illegible] the identity

$$r^*(\mathbf{n}')r(\mathbf{n}') + t^*(\mathbf{n}')t(\mathbf{n}') = 1. \quad (5.9)$$

[illegible] evidently expresses *conservation of energy*.

[illegible] the help of the reciprocity relation (3.6b) the formu[illegible] may readily be expressed in the form of a Stokes [illegible] for this purpose, and [illegible] connection with a derivation of another Stokes relation [illegible] were the incident wave propagat[illegible] the direction specified [illegible] would have, by analogy with Eq. [illegible]

$$[illegible] \quad (5.10a)$$

$$[illegible] \quad (5.10b)$$

Here [illegible] transmission coefficients, respectively, [illegible] the half-space [illegible] and $\mathbf{n}_r(\mathbf{n})$ and [illegible] directions of the reflected and transmitted [illegible], which may be determined [illegible], respectively.

We have [illegible]

$$[illegible] \quad (5.11)$$

Now since the stratified [illegible] to be surrounded on both sides by free space [illegible] refraction for stratified media [Ref. [illegible] Eq. (24)] gives at once $n_t(-\mathbf{n}) = -\mathbf{n}$, and the form [illegible] becomes

$$t(-\mathbf{n}', -\mathbf{n}) = t(-\mathbf{n})\Delta(-\mathbf{n}' + \mathbf{n}). \quad (5.12)$$

It also follows from Eq. [illegible] and the relations $n_t(\mathbf{n}') = \mathbf{n}'$ and $\mathbf{n}(n'') = \mathbf{n}$ (which [illegible] at once from the law of refraction) that

$$[illegible] \quad (5.13)$$

On substituting [illegible] Eqs. [illegible] and (5.12) into the reciprocity relation (3.6b) and integrating over the unit sphere generated by the vector $\mathbf{n}$, we find that

$$t(-\mathbf{n}) = t(\mathbf{n}). \quad (5.14)$$

This formula obviously expresses [illegible] *for transmission by a stratified medium*.

On substituting from Eq. [illegible] into Eq. (5.9) we obtain the relation

$$[illegible] \quad (5.15)$$

This [illegible] *for stratified dielectric media* [illegible] well known in the theory of [illegible]

B. Consequences of Eq. (4.2b)

Next we consider the implications of the generalized Stokes relation (4.2b). Suppose that the plane wave is incident upon the [illegible] medium [illegible] in a direction specified [illegible]. On substituting from Eqs. (5.10b) and [illegible] first integral on the left-hand sides of Eqs. (4.2b) [illegible] that

$$\int t^*(\mathbf{n}, \mathbf{n}')r(\mathbf{n}, \mathbf{n}'')d\Omega$$

$$= t^*(\mathbf{n}')r(\mathbf{n}'') \int [illegible] \Delta[\mathbf{n} - \mathbf{n}_r(\mathbf{n}'')]d\Omega. \quad (5.16)$$

The integral [illegible] readily be evaluated by the use of Eq. (C3) of Appendix C and the fact that $n_z < 0$. If we also make use of the formula [illegible] (which follows at once from the law of refraction [illegible] media) and recall Eq. (5.3), we find that

$$\int_{n'} r^*(\mathbf{n},\mathbf{n}')r(\mathbf{n},\mathbf{n}'')\mathrm{d}\Omega$$

$$= \tilde{r}^*(\mathbf{n}')\tilde{r}(\mathbf{n}'')\frac{\delta[\theta'-(\pi-\theta'')]\delta(\varphi'-\varphi'')}{|\sin\theta'|}. \quad (5.17)$$

In a similar way one may evaluate the second integral on the right-hand side of Eq. (5.16), also making use of the fact that with $\mathbf{n}' = (\theta', \varphi')$, $\mathbf{n}_r(\mathbf{n}') = (\pi - \theta', \varphi')$. One then readily finds that

$$\int_{n'} \rho^*(\mathbf{n},\mathbf{n}')t(\mathbf{n},\mathbf{n}'')\mathrm{d}\Omega$$

$$= \tilde{\rho}^*(\mathbf{n}')\tilde{t}(\mathbf{n}'')\frac{\delta(\theta'+\theta''-\pi)\delta(\varphi'-\varphi'')}{|\sin\theta'|}. \quad (5.18)$$

On substituting from Eqs. (5.17) and (5.18) into the generalized Stokes relation (4.2b), integrating both sides of the equation over the unit sphere generated by the unit vector $\mathbf{n}' = (\theta', \varphi')$, and making use of Eq. (5.3), we find that

$$\tilde{r}^*(\mathbf{n}_r'')\tilde{r}(\mathbf{n}'') + \tilde{\rho}^*(\mathbf{n}_r'')\tilde{t}(\mathbf{n}'') = 0, \quad (5.19)$$

where $\mathbf{n}_r'' = \mathbf{n}_r(\mathbf{n}'')$ is the unit vector along the direction of the reflected wave when the incident wave propagates in the direction specified by the unit vector $\mathbf{n}''$ ($n_z'' > 0$).

If in Eq. (5.19) we make use of the reciprocity relation (5.14) and write $\mathbf{n}'$ in place of $\mathbf{n}''$, we obtain the formula

$$\tilde{t}^*(-\mathbf{n}_r')\tilde{r}(\mathbf{n}') + \tilde{\rho}^*(\mathbf{n}_r')\tilde{t}(\mathbf{n}') = 0. \quad (5.20)$$

There are other forms in which the relations (5.19) and (5.20) can be expressed. For example, since the unit vectors $\mathbf{n}'$ and $-\mathbf{n}_r'$ make the same angle with the z axis, $\tilde{t}(-\mathbf{n}_r') = \tilde{t}(\mathbf{n}')$. For the same reason $\tilde{\rho}(\mathbf{n}_r') = \tilde{\rho}(-\mathbf{n}')$. Making use of these relations in Eq. (5.20), we obtain the formula

$$\tilde{t}^*(\mathbf{n}')\tilde{r}(\mathbf{n}') + \tilde{t}(\mathbf{n}')\tilde{\rho}^*(-\mathbf{n}') = 0. \quad (5.21)$$

This formula is *another Stokes relation for stratified dielectric media* and is of a form well known in the theory of dielectric films (Ref. 2, p. 173).

The relations (5.21) and (5.15) recently played an important role in the theory of correction of distortions by the technique of phase conjugation.[4]

APPENDIX A: DERIVATION OF A FORMAL ASYMPTOTIC APPROXIMATION

We begin by recalling that under very general conditions any solution $V(\mathbf{r})$, valid throughout the whole space, of the Helmholtz equation

$$\nabla^2 V(\mathbf{r}) + k^2 V(\mathbf{r}) = 0 \quad (A1)$$

may be expressed in the form of an angular spectrum of homogeneous plane waves, all with the same wave number k, that propagate in all possible directions[13]:

$$V(\mathbf{r}) = \int_{\Omega} a(\mathbf{n})e^{ik\mathbf{n}\cdot\mathbf{r}}\mathrm{d}\Omega. \quad (A2)$$

The complex spectral amplitude function $a(\mathbf{n})$ can be derived from the knowledge of $V(\mathbf{r})$ by the inversion formula [Ref. 13, Eq. (B14)]

$$a(\mathbf{n}) = \frac{k^2}{(2\pi)^3}\lim_{\epsilon\to+0}\int_{k-\epsilon}^{k+\epsilon}\mathrm{d}K\int V(\mathbf{r})e^{-iK\mathbf{n}\cdot\mathbf{r}}\mathrm{d}^3r. \quad (A3)$$

The asymptotic behavior of $V(r\mathbf{u})$ as $kr \to \infty$ in the direction of a real unit vector $\mathbf{u}$ may be obtained by the use of Eqs. (2.1) and formula (2.3) or, somewhat more directly, from a mathematical lemma due to Jones.[14] The result is

$$V(r\mathbf{u}) \sim \frac{2\pi}{ik}\left[a(\mathbf{u})\frac{e^{ikr}}{r} - a(-\mathbf{u})\frac{e^{-ikr}}{r}\right] \quad \text{as } kr \to \infty. \quad (A4)$$

Let us now apply these results to the case when $V(\mathbf{r})$ is a plane wave of unit amplitude that propagates in the direction of a unit vector $\mathbf{n}_0$:

$$V(\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}. \quad (A5)$$

To determine the angular spectrum amplitude function $a(\mathbf{n})$ of this field we first note that, when $V(\mathbf{r})$ is given by Eq. (A5), the Fourier transform that appears in Eq.(A3) becomes

$$\int V(\mathbf{r})e^{-iK\mathbf{n}\cdot\mathbf{r}}\mathrm{d}^3r = \int \exp[i(k\mathbf{n}_0 - K\mathbf{n})\cdot\mathbf{r}]\mathrm{d}^3r$$

$$= (2\pi)^3\delta^{(3)}(k\mathbf{n}_0 - K\mathbf{n}), \quad (A6)$$

where $\delta^{(3)}$ is, of course, the three-dimensional Dirac delta function. On substituting from Eq. (A6) into Eq. (A3) we find that $a(\mathbf{n})$ is now given by the formula

$$a(\mathbf{n}) = k^2\lim_{\epsilon\to+0}\int_{k-\epsilon}^{k+\epsilon}\delta^{(3)}(k\mathbf{n}_0 - K\mathbf{n})\mathrm{d}K. \quad (A7)$$

To evaluate the integral on the right-hand side of Eq. (A7) we make use of the representation of the three-dimensional Dirac delta function in spherical polar coordinates.[15] One then finds at once that

$$\delta^{(3)}(k\mathbf{n}_0 - K\mathbf{n}) = (1/k^2)\Delta(\mathbf{n} - \mathbf{n}_0)\delta(k - K), \quad (A8)$$

where Δ is defined by Eq. (2.12) and $\delta(k - K)$ is the one-dimensional Dirac delta function. On substituting from Eq. (A8) into Eq. (A7) and carrying out the trivial integration with respect to K, we find that the angular spectrum amplitude function of the plane-wave field is simply

$$a(\mathbf{n}) = \Delta(\mathbf{n} - \mathbf{n}_0). \quad (A9)$$

Finally, on substituting from Eqs. (A5) and (A9) into the asymptotic formula (A4) we obtain the formal asymptotic approximation

$$e^{ik\mathbf{n}_0\cdot\mathbf{r}} \sim \frac{2\pi}{ik}\left[\Delta(\mathbf{n} - \mathbf{n}_0)\frac{e^{ikr}}{r} - \Delta(\mathbf{n} + \mathbf{n}_0)\frac{e^{-ikr}}{r}\right] \quad \text{as } kr \to \infty. \quad (A10)$$

It is of interest to note the form of the angular spectrum representation of a plane wave. It is clear on comparing the right-hand sides of Eqs. (A10) and (2.3) that for the plane wave defined by Eq. (A5)

$$C^{(+)}(\mathbf{n}) = -\frac{2\pi}{ik}\Delta(\mathbf{n} - \mathbf{n}_0), \quad (A11a)$$

$$D^{(+)}(\mathbf{n}) = \frac{2\pi}{ik}\Delta(\mathbf{n} - \mathbf{n}_0). \quad (A11b)$$

On taking in (Eq. 2.1) $U^i(\mathbf{r}) = \exp(ik\mathbf{n}_0\cdot\mathbf{r})$ and on substituting for the C and D coefficients the expressions (A11), we find that

In $\mathscr{R}^-$:

$$e^{ik\mathbf{n}_0\cdot\mathbf{r}} = \int_{\sigma^{(+)}} \Delta(\mathbf{n} - \mathbf{n}_0)e^{ik\mathbf{n}\cdot\mathbf{r}}d\Omega + \int_{\sigma^{(-)}} \Delta(\mathbf{n} - \mathbf{n}_0)e^{ik\mathbf{n}\cdot\mathbf{r}}d\Omega, \quad \text{(A12a)}$$

In $\mathscr{R}^+$:

$$e^{ik\mathbf{n}_0\cdot\mathbf{r}} = \int_{\sigma^{(-)}} \Delta(\mathbf{n} - \mathbf{n}_0)e^{ik\mathbf{n}\cdot\mathbf{r}}d\Omega + \int_{\sigma^{(+)}} \Delta(\mathbf{n} - \mathbf{n}_0)e^{ik\mathbf{n}\cdot\mathbf{r}}d\Omega. \quad \text{(A12b)}$$

One can verify by direct evaluations of the integrals on the right-hand sides of these equations that these formulas hold throughout a wider domain than indicated here; in fact, each of the two Eqs. (A12) is a valid representation of the plane wave $\exp(-ik\mathbf{n}_0\cdot\mathbf{r})$ throughout the whole space.

APPENDIX B: DERIVATION OF FORMULAS (2.15)

Suppose that the field incident upon the scatterer is a plane wave of unit amplitude that propagates in the direction of a unit vector $\mathbf{n}_0$:

$$U^{(i)}(\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}. \quad \text{(B1)}$$

The total field (incident + scattered) in the far zone is given by a formula of the form

$$U(r\mathbf{n}) \sim e^{ik\mathbf{n}_0\cdot\mathbf{r}} + A(\mathbf{n}, \mathbf{n}_0)\frac{e^{ikr}}{r}, \quad \text{as } kr \to \infty, \quad \text{(B2)}$$

where $A(\mathbf{n}, \mathbf{n}_0)$ is the scattering amplitude. If we substitute in Eq. (B2) for $\exp(ik\mathbf{n}_0\cdot\mathbf{r})$ its formal asymptotic approximation given by formula (A10), the expression (B2) for $U(r\mathbf{n})$ acquires the form (2.9), viz.,

$$U(r\mathbf{n}) \sim F_1(\mathbf{n})\frac{e^{ikr}}{r} + F_2(\mathbf{n})\frac{e^{-ikr}}{r}, \quad \text{as } kr \to \infty, \quad \text{(B3)}$$

where

$$F_1(\mathbf{n}) = \frac{2\pi}{ik}\Delta(\mathbf{n} - \mathbf{n}_0) + A(\mathbf{n}, \mathbf{n}_0), \quad \text{(B4a)}$$

$$F_2(\mathbf{n}) = -\frac{2\pi}{ik}\Delta(\mathbf{n} + \mathbf{n}_0). \quad \text{(B4b)}$$

The expression (B4a) may readily be expressed in terms of the S matrix. To do so, we substitute from Eqs. (B4) into the formula (2.10) that may be regarded as a definition of the S matrix. We then find, after trivial calculation, that

$$A(\mathbf{n}, \mathbf{n}_0) = \frac{2\pi}{ik}[\mathscr{S}(\mathbf{n}, \mathbf{n}_0) - \Delta(\mathbf{n} - \mathbf{n}_0)]. \quad \text{(B5)}$$

On comparing Eqs. (B5) and (B4a) we see at once that

$$F_1(\mathbf{n}) = \frac{2\pi}{ik}\mathscr{S}(\mathbf{n}, \mathbf{n}_0). \quad \text{(B6)}$$

The formulas (B6) and (B4b) are Eqs. (2.15) of the text.

APPENDIX C: DERIVATION OF THE FORMULA $\int_{4\pi} \Delta(\mathbf{n} - \mathbf{n}')\Delta(\mathbf{n} - \mathbf{n}'')d\Omega = \Delta(\mathbf{n}' - \mathbf{n}'')$

We have, according to the definition (2.12) of the "spherical" Dirac delta function

$$\Delta(\mathbf{n} - \mathbf{n}') = \frac{\delta(\theta - \theta')\delta(\varphi - \varphi')}{|\sin\theta'|}, \quad \text{(C1)}$$

where (θ, φ) and (θ', φ') are the spherical polar angles of the unit vectors $\mathbf{n}$ and $\mathbf{n}'$, respectively. We have a similar expression for $\Delta(\mathbf{n} - \mathbf{n}'')$. Hence it follows that

$$\int_{(4\pi)} \Delta(\mathbf{n} - \mathbf{n}')\Delta(\mathbf{n} - \mathbf{n}'')d\Omega = \frac{1}{|\sin\theta'|}\int_0^\pi\int_0^{2\pi} \frac{\delta(\theta - \theta')\delta(\varphi - \varphi')\delta(\theta - \theta'')\delta(\varphi - \varphi'')}{|\sin\theta'|}\sin\theta d\theta d\varphi. \quad \text{(C2)}$$

Now for $0 \le \theta \le \pi$, $|\sin\theta| = \sin\theta$, and Eq. (C2) therefore reduces to

$$\int_{(4\pi)} \Delta(\mathbf{n} - \mathbf{n}')\Delta(\mathbf{n} - \mathbf{n}'')d\Omega = \int_0^\pi \delta(\theta - \theta')\delta(\theta - \theta'')d\theta \times \int_0^{2\pi} \delta(\varphi - \varphi')\delta(\varphi - \varphi'')d\varphi. \quad \text{(C3)}$$

By an elementary property of the Dirac delta function [Ref. 12, App. IV, Eq. (12)] the first integral on the right-hand side is equal to $\delta(\theta' - \theta'')$ and the second to $\delta(\varphi' - \varphi'')$. Using these facts, Eq. (C2) reduces to

$$\int_{(4\pi)} \Delta(\mathbf{n} - \mathbf{n}')\Delta(\mathbf{n} - \mathbf{n}'')d\Omega = \frac{\delta(\theta' - \theta'')\delta(\varphi' - \varphi'')}{|\sin\theta'|} \quad \text{(C4)}$$

or, recalling again the definition of the "spherical" Dirac delta function [see Eq. (C1)],

$$\int_{(4\pi)} \Delta(\mathbf{n} - \mathbf{n}')\Delta(\mathbf{n} - \mathbf{n}'')d\Omega = \Delta(\mathbf{n}' - \mathbf{n}''). \quad \text{(C5)}$$

ACKNOWLEDGMENTS

This research was supported by the National Science Foundation and the U.S. Air Force Geophysics Laboratory under AFOSR Task 2310G1.

We are indebted to J. T. Foley for some helpful suggestions relating to the calculations in Section 5.

Emil Wolf is also with the Institute of Optics, University of Rochester.

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$$U^{(i)}(\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}$$

formally has the asymptotic behavior (see Appendix A)

$$e^{ik\mathbf{n}_0\cdot\mathbf{r}} \sim \frac{2\pi}{ik}\left[\Delta(\mathbf{n}-\mathbf{n}_0)\frac{e^{ikr}}{r} - \Delta(\mathbf{n}+\mathbf{n}_0)\frac{e^{-ikr}}{r}\right]$$

as $kr \to \infty$, with the unit vector $\mathbf{n}$ fixed, and Δ is the "spherical" delta function, defined by Eq. (2.12). Hence a plane wave provides both incoming and outgoing contributions at infinity.
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OPTICA ACTA, 1986, VOL. 33, NO. 11, 1369–1376

Radiance theorem with partially coherent light

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(*Received 20 January 1986; revision received 30 July 1986*)

Abstract. The transmission of a generalized radiance across a planar boundary separating two homogeneous media is considered. It is assumed that the optical field remains continuous at the interface and reflection is neglected. A result is obtained which may be regarded as a generalization of the conventional radiance theorem for fields of any state of coherence. This result differs from the conventional theorem by a factor that depends, in general, both on the optical intensity and on the degree of coherence of the field. However, over a wide range of circumstances the generalized radiance theorem is shown to be in good agreement with the conventional theorem.

1. Introduction

One of the basic principles of conventional radiometry [1] is the so-called radiance (or brightness) theorem that pertains, in its most general form, to the relationship between the radiance of an object and the radiance of its image formed by any specular optical system.† Within the framework of linear theory, an arbitrarily complicated specular optical system may be considered simply as being composed of a sequence of uniform media separated by sharp boundaries. The conventional radiance theorem then follows directly from the phenomenological laws that govern the transmission of the radiance through a uniform medium and across a boundary separating two uniform media with different indices of refraction.

The propagation of the conventional radiance is governed, under general circumstances, by the equation of radiative transfer ([2], chapter 1, equation (47)). It implies that in a uniform medium (that does not contain sources or absorbers) the radiance function $B_\omega(\mathbf{r}, \mathbf{s})$ at some frequency ω, measured in the direction specified by the unit vector $\mathbf{s}$, remains invariant on the line in the direction $\mathbf{s}$ through the point represented by the vector $\mathbf{r}$. In a number of recent publications (see, for example, [3–8], the validity of the equation of radiative transfer has been investigated in a (statistically) homogeneous medium with scalar fields of arbitrary states of coherence. The discussion has also been extended into the domain of electromagnetic fields both within the framework of classical [9, 10] and quantized [11, 12] wave theories.

† A specular optical system in this context is one that does not contain diffusely transmitting (or reflecting) surfaces.

The transformation of the radiance across a sharp boundary has, however, received considerably less attention. Although the reflection and refraction of wave fields at boundaries have been extensively studied, no work is known to the author that deals specifically with the transmission of the radiance associated with a fluctuating optical field through a medium discontinuity surface.† For this reason we concentrate in this paper on examining the conventional radiance theorem and its range of validity with partially coherent light at a single planar boundary between two homogeneous media.

In the present paper we will adopt a relatively simple and straightforward approach that is based on the scalar theory of light. The fundamental assumption, as is customary in physical optics, is that the (monochromatic) optical field remains continuous across the interface. Nonetheless, the method employed takes into account some interesting physical phenomena such as the conversion of evanescent waves into propagating plane waves. Moreover, it offers several valuable clues to a future improvement of the analysis.

2. Radiance theorem of conventional radiometry

We begin by briefly recalling the phenomenological form of the radiance theorem as it is traditionally encountered in radiometry. Conventional radiometry deals with the problem of energy transport at some temporal frequency ω. With reference to figure 1, the conventional radiance theorem at a single refracting surface may be expressed in the form [14]

$$\frac{B_1(\mathbf{r},\mathbf{s}_1)}{n_1^2}=\frac{B_2(\mathbf{r},\mathbf{s}_2)}{n_2^2}, \tag{1}$$

where B_1 and B_2 are the values of the radiance on the two sides of the interface (the explicit ω dependence is omitted), n_1 and n_2 are the refractive indices of the two uniform media, and $\mathbf{r}$ denotes the position vector of an element $d\sigma$ of the boundary. Further, it is important to note that in the conventional radiance theorem (1) $\mathbf{s}_1$ and $\mathbf{s}_2$ are unit vectors that specify the path of a geometrical light ray across the surface element. The effects of reflection have been neglected in the derivation of equation (1).

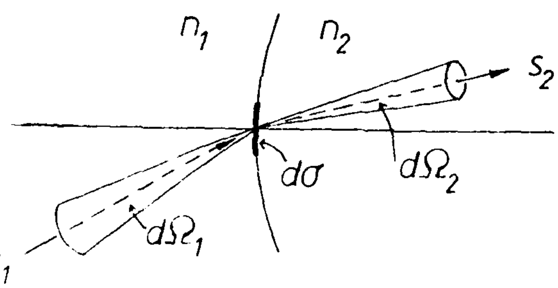

Figure 1. Illustration of the notation relating to the radiance theorem of conventional radiometry.

† The propagation of generalized radiance functions (that pertain to fields of any state of coherence) in lens systems has been studied in [4] and [8] on the basis of the usual Fourier optics approximations. Also, a rather general (one-dimensional) analysis of the propagation of a generalized radiance through lenses was presented in [13] making use of asymptotic approximations (geometrical optics limit) based on the principle of the stationary phase.

The radiance theorem (1) of conventional radiometry expresses merely the conservation of energy (at frequency ω) that is incident on the element of $d\sigma$ from the differential solid angle $d\Omega_1$ around the direction $\mathbf{s}_1$ and emerges into the differential solid angle $d\Omega_2$ around the direction $\mathbf{s}_2$ (see figure 1). Therefore, it is clear that in connection with the conventional radiance theorem, the size and direction of the element $d\Omega_2$ are directly determined by $d\Omega_1$ through Snell's law of refraction.

The losses due to reflection could be included in the conventional radiance theorem (1) by introducing a phenomenological coefficient of reflection that depends on both position and direction. Formally this would involve the use of the differential scattering coefficient ([2], chapter 1, § 3) that appears in the conventional equation of radiative transfer. Approximate values for the reflection coefficient could be obtained, for example, from measurements or from the customary Fresnel equations (see, for example, [15]).

3. Generalized radiance and the radiance theorem with light of any state of coherence

In the context of fluctuating optical fields, geometrical optics cannot be used to couple the energy transport on the two sides of the boundary. For this reason we will make the assumptions, common in physical optics [16], that the optical field remains continuous in passing across the boundary surface and that on the surface it is given simply by the incident field. These assumptions are the cornerstones of the customary analysis of scalar-wave propagation in optical systems, where the various elements such as lenses are represented by complex-amplitude transmission functions. Clearly, the assumed field properties then also imply that reflection at the interface has been neglected.† Since the continuity holds for each realization of the statistical ensemble (assumed to be stationary), the cross-spectral density function [18] $W(\mathbf{r}_1, \mathbf{r}_2)$ that characterizes the spatial coherence properties of the field at frequency ω, will also be continuous across the boundary.

For the sake of simplicity, we take the refracting surface separating the two homogeneous media to be a plane $z = \text{constant}$, say $z = z_0$, and consider a wavefield propagating across the boundary into the half-space $z > z_0$ (figure 2). We may then associate with the field distribution in any transverse plane $z = \text{constant}$ a generalized radiance function defined by the expression‡ ([19], equation (21))

$$B(\boldsymbol{\rho}, \mathbf{s}_\perp) = (k/2\pi)^2 \cos\theta \int W(\boldsymbol{\rho} + 1/2\boldsymbol{\rho}', \boldsymbol{\rho} - 1/2\boldsymbol{\rho}') \exp(-ik\mathbf{s}_\perp \cdot \boldsymbol{\rho}')\, d^2\rho', \tag{2}$$

where $W(\boldsymbol{\rho} + 1/2\boldsymbol{\rho}', \boldsymbol{\rho} - 1/2\boldsymbol{\rho}')$ denotes the cross-spectral density (at frequency ω) of the light at the points $\boldsymbol{\rho}_1' = \boldsymbol{\rho} + 1/2\boldsymbol{\rho}'$ and $\boldsymbol{\rho}_2 = \boldsymbol{\rho} - 1/2\boldsymbol{\rho}'$ in that plane, and

$$k = nk_0 = n(\omega/c), \tag{3}$$

† In scalar optics one sometimes requires that both the optical field and its normal derivative remain continuous across a sharp boundary. These boundary conditions then give rise also to a reflected field component. In particular, with a planar boundary and an incident plane wave, the resulting coefficients for reflection and refraction are the usual Fresnel equations for the case when the electric field is perpendicular to the plane of incidence (compare, for example, [17], equations (5) and (6), and [15], equations (4.34) and (4.35)).

‡ Since the Cartesian components s_x, s_y and s_z of the unit vector $\mathbf{s}$ are related by the identity $s_x^2 + s_y^2 + s_z^2 = 1$, only two of the three components are independent. We will, therefore, regard the generalized radiance as being, in its directional dependence, a function of the two-dimensional transverse vector $\mathbf{s}_\perp = (s_x, s_y)$ and denote the radiance by $B(\boldsymbol{\rho}, \mathbf{s}_\perp)$.

with n being the refractive index of the medium and c the speed of light in vacuum. Further, in equation (2), θ denotes the angle between the unit vector $\mathbf{s}$ and the positive z axis and $\mathbf{s}_\perp$ is the projection of $\mathbf{s}$ (considered as a two-dimensional vector) onto the plane z = constant, i.e. if $\mathbf{s} = (s_x, s_y, s_z)$, then $\mathbf{s}_\perp = (s_x, s_y, 0)$. Finally, the integration in equation (2) extends throughout the entire plane z = constant.

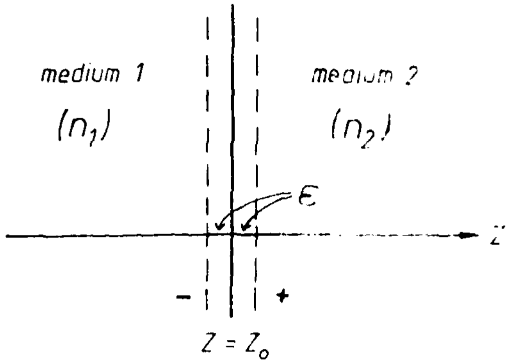

Figure 2. A planar boundary $z = z_0$ separating two homogeneous media with refractive indices n_1 and n_2, respectively.

For later analysis it will be convenient to introduce an auxiliary quantity, known as the Wigner (distribution) function [20], that is closely related to the symmetrical definition of the generalized radiance given in equation (2). We define the Wigner function by the formula

$$\chi(\boldsymbol{\rho}, \mathbf{f}) = \int W(\boldsymbol{\rho} + 1/2\boldsymbol{\rho}', \boldsymbol{\rho} - 1/2\boldsymbol{\rho}') \exp(-i\mathbf{f} \cdot \boldsymbol{\rho}')\, d^2\rho' \qquad (4)$$

where $\mathbf{f}$ is a (real) two-dimensional vector. An analogous quantity, which can be identified [21] in some sense with the 'local spatial-frequency spectrum' of the field, has been studied extensively by Bastiaans [22, 23] in geometrical and coherent physical optics.

On comparing equations (2) and (4), we see at once that

$$B(\boldsymbol{\rho}, \mathbf{s}_\perp) = (k/2\pi)^2 \cos\theta\, \chi(\boldsymbol{\rho}, k\mathbf{s}_\perp) \qquad (5)$$

a relation which is valid in any transverse plane z = constant. We will consider, in particular, two planes displaced by a small distance ε from the plane $z = z_0$ in media 1 (refractive index n_1) and 2 (index n_2), and denote the generalized radiance functions in these planes by subscripts − and +, respectively (see figure 2). Making use of the facts that there are no backward-propagating fields (reflection is neglected) and that the cross-spectral density function $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ remains continuous across the boundary, we then obtain, in the limit as $\varepsilon \to 0$, from equations (3)–(5) the relation

$$B_+(\boldsymbol{\rho}, \mathbf{s}_{2\perp}) = (n_2/n_1)^2 M(\boldsymbol{\rho}; \mathbf{s}_{1\perp}, \mathbf{s}_{2\perp}) B_-(\boldsymbol{\rho}, \mathbf{s}_{1\perp}) \qquad (6)$$

where

$$M(\boldsymbol{\rho}; \mathbf{s}_{1\perp}, \mathbf{s}_{2\perp}) = \frac{\cos\theta_2\, \chi(\boldsymbol{\rho}, k_2\mathbf{s}_{2\perp})}{\cos\theta_1\, \chi(\boldsymbol{\rho}, k_1\mathbf{s}_{1\perp})} \qquad (7)$$

In this expression, $k_1 = n_1 k_0$ and $k_2 = n_2 k_0$ with $k_0 = \omega/c$, as before, and θ_1 and θ_2 are the angles that the directions $\mathbf{s}_1$ and $\mathbf{s}_2$ make with the positive z axis.

Formula (6) may now be regarded as a generalization (at a planar boundary) of the conventional radiance theorem (1) for optical wavefields of any state of coherence. It is, in essence, an identity that follows directly from the basic assumptions of physical optics. Comparison of equations (1) and (6) reveals that the new relation (6) contains an additional factor $M(\boldsymbol{\rho}; \mathbf{s}_{1\perp}, \mathbf{s}_{2\perp})$, which is given by equation (7). This factor, determined primarily by the state of coherence of the optical field at the boundary, is a measure of the extent to which the present generalized result differs from the conventional law connecting the radiances B_1 and B_2. Through the cross-spectral density appearing in the definition (4) of the Wigner function χ, the factor M depends, in general, both on the optical intensity and on the complex degree of spatial coherence of the light at the interface. In fact, we see from equation (7) that, apart from a purely geometrical part, the factor M is simply the ratio of the values that the associated function $\chi(\boldsymbol{\rho}, \mathbf{f})$ assumes with the arguments $\mathbf{f}_2 = k_2 \mathbf{s}_{2\perp}$ and $\mathbf{f}_1 = k_1 \mathbf{s}_{1\perp}$.

If the directional vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ specify a geometrical ray path across the boundary, then $k_2 \mathbf{s}_{2\perp} = k_1 \mathbf{s}_{1\perp}$ according to Snell's law and the additional factor M reduces in this case to the ratio $\cos\theta_2/\cos\theta_1$. This result is a consequence of the requirements that there is no reflected wave corresponding to a wave incident from the direction $\mathbf{s}_1$ and that the transmitted wave in the direction $\mathbf{s}_2$ matches the values of the incident wave at the interface. The result holds separately for any incident-wave direction and also implies that under the present assumptions energy is not strictly conserved in passage across the surface. However, if the appropriate reflection and transmission coefficients are included, the energy conservation for plane waves is restored. Conversely, straightforward calculations using the general results (6) and (7) show that if $B_-(\boldsymbol{\rho}, \mathbf{s}_{1\perp})$ is zero except for some value $\mathbf{s}_{0\perp}$, then $B_+(\boldsymbol{\rho}, \mathbf{s}_{2\perp})$ will also differ from zero only when $n_2 \mathbf{s}_{2\perp} = n_1 \mathbf{s}_{0\perp}$, in accordance with geometrical optics.

We will emphasize, furthermore, that unlike in the conventional radiance theorem (1), the variables $\mathbf{s}_{1\perp}$ and $\mathbf{s}_{2\perp}$ in the generalized result expressed by equations (6) and (7) are projections of quite arbitrary unit vectors that point towards the half-space $z > 0$. This makes it possible to use the analytic properties of the generalized radiance function $B(\boldsymbol{\rho}, \mathbf{s}_\perp)$. It implies also, for example, that in the case when $n_1 < n_2$, the generalized radiance $B_+(\boldsymbol{\rho}, \mathbf{s}_{2\perp})$ may be non-zero even in the domain $n_1/n_2 < |\mathbf{s}_{2\perp}| < 1$, corresponding to angles θ_2 larger than the critical angle of total internal reflection. Physically, such a situation represents the phenomenon where evanescent waves are turned into homogeneous (propagating) waves by refraction at the discontinuity.

4. Radiance theorem with quasi-homogeneous light

Let us assume now that the optical field at the interface is quasi-homogeneous, i.e. one that is characterized by a cross-spectral density function of the form [24]

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = I((\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)/2)\, g(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2), \tag{8}$$

where $I(\boldsymbol{\rho})$, the optical intensity, is a 'slow' function of $\boldsymbol{\rho}$ and $g(\boldsymbol{\rho}')$, the complex degree of spatial coherence [18], is a 'fast' function of $\boldsymbol{\rho}'$ (see [24], § 11). The Wigner function associated with such a field distribution is readily found from equation (4) to be given by the expression

$$\gamma(\boldsymbol{\rho}, \mathbf{f}) = (2\pi)^2 I(\boldsymbol{\rho})\tilde{g}(\mathbf{f}), \tag{9}$$

where

$$\tilde{g}(\mathbf{f}) = (1/2\pi) \int g(\boldsymbol{\rho}') \exp(-i\mathbf{f}\cdot\boldsymbol{\rho}')\, d^2\rho' \tag{10}$$

is the two-dimensional spatial Fourier transform of $g(\boldsymbol{\rho}')$. On substituting from equation (9) into the general formula (7), we obtain the expression

$$M(\boldsymbol{\rho}; \mathbf{s}_{1\perp}, \mathbf{s}_{2\perp}) = \frac{\cos\theta_2\, \tilde{g}(k_2\mathbf{s}_{2\perp})}{\cos\theta_1\, \tilde{g}(k_1\mathbf{s}_{1\perp})}. \tag{11}$$

Equation (11) shows that for a quasi-homogeneous field the factor M, which is absent in the usual formula (1), is independent of the optical intensity of the light distribution at the interface. Moreover, since $g(\boldsymbol{\rho}')$ is a 'fast' function of $\boldsymbol{\rho}'$, its Fourier transform $\tilde{g}(\mathbf{f})$ is a 'slow' function of $\mathbf{f}$. Consequently for quasi-homogeneous light $M(\boldsymbol{\rho}; \mathbf{s}_{1\perp}, \mathbf{s}_{2\perp}) \approx 1$ and the generalized result (6) is seen to approximate the conventional radiance theorem (1) with relatively good accuracy over a range of directions $\mathbf{s}_1$ and $\mathbf{s}_2$ such that $|\mathbf{s}_{1\perp}| \approx |\mathbf{s}_{2\perp}|$. We note briefly also that for statistically homogeneous fields equation (11) remains valid even when $g(\boldsymbol{\rho}')$ is not a sharply peaked function. For such fields the generalized result (6) is seen to depend on the functional form of the complex degree of spatial coherence $g(\boldsymbol{\rho}')$.

Let us now assume, furthermore, that $n_1 < n_2$ and that the complex degree of spatial coherence of the light at the interface is given by the expression

$$g(\boldsymbol{\rho}') = \frac{\sin k_1\rho'}{k_1\rho'}, \tag{12}$$

where $\rho' = |\boldsymbol{\rho}'|$. This expression is characteristic of a Lambertian radiator, such as blackbody radiation source [25]. Making use of the definition (2), the generalized radiance $B_1(\boldsymbol{\rho}, \mathbf{s}_{1\perp})$ is then found to be independent of the directional variable $\mathbf{s}_{1\perp}$, and we will denote it by $B_0(\boldsymbol{\rho})$. On substituting from equation (12) into equation (11) and making use of formula (6), we obtain the following result:

$$\begin{aligned} B_t(\boldsymbol{\rho}, \mathbf{s}_{2\perp}) &= \left(\frac{n_2}{n_1}\right)^2 \left[\frac{\cos^2\theta_2}{1-(n_2/n_1)^2\sin^2\theta_2}\right]^{1/2} B_0(\boldsymbol{\rho}) \quad &\text{if } \theta_2 < \theta_c, \\ &= 0 &\text{if } \theta_2 > \theta_c. \end{aligned} \tag{13}$$

Here the angle θ_c is defined by the relation

$$\sin\theta_c = n_1/n_2. \tag{14}$$

The formula (13) shows that if the field at the interface is quasi-homogeneous (or strictly homogeneous) with its complex degree of spatial coherence given by equation (12), then there is a maximum angle, θ_c, beyond which no energy is transmitted.† According to equation (14), this angle is precisely the critical angle of total internal reflection.

† This result is a consequence of the fact that the spatial Fourier transform of the correlation function (12) is identically zero outside the domain $|\mathbf{f}| < k_1$ (see [25], § 11). Hence the field incident on the boundary contains no evanescent waves that could be turned by the discontinuity surface into homogeneous waves propagating at angles larger than θ_c.

Formula (13) also implies that even for angles less than the maximum angle θ_c, there is strictly speaking an angular dependence that is not present in the conventional form (1) of the radiance theorem. This angular dependence is a consequence of the basic assumptions of physical optics (compare the discussion following equations (6) and (7)), and it is illustrated in figure 3 for two values of the ratio n_2/n_1. These curves, calculated according to equation (13), indicate that the ratio $B_+(\boldsymbol{\rho}, \mathbf{s}_{21})/B_0(\boldsymbol{\rho})$ remains substantially constant over a relatively wide range of angles θ_2, in agreement with the conventional radiance theorem. Such a behaviour becomes even more dominant as the ratio n_2/n_1 is decreased. In the limit as n_2/n_1 approaches unity, there is no refracting surface and $B_+(\boldsymbol{\rho}, \mathbf{s}_{11})$ becomes, of course, identical to $B_0(\boldsymbol{\rho})$.

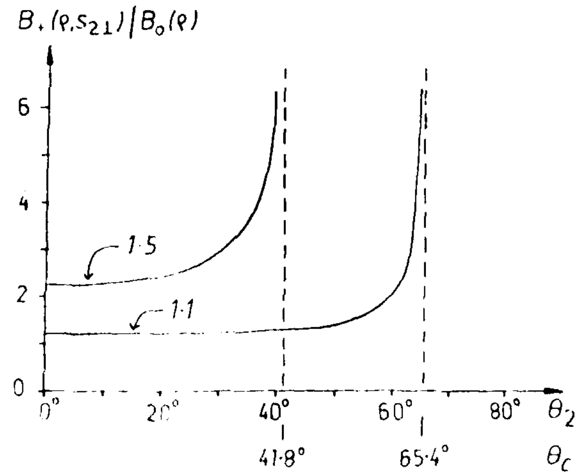

Figure 3. Dependence of $B_+(\boldsymbol{\rho}, \mathbf{s}_{21})/B_0(\boldsymbol{\rho})$ on the angle θ_2 ($\sin\theta_2 = |\mathbf{s}_{21}|$) for two values of the ratio n_2/n_1, namely 1·1 and 1·5, when the degree of field correlation at the interface is given by equation (12). The angle θ_c denotes the maximum angle beyond which no energy is transmitted.

5. Summary and discussion

In this paper we studied the radiance theorem in the context of partially coherent waves and considered only refraction at a planar interface separating two homogeneous media. The analysis was carried out within the framework of the scalar theory of light. It was based on the assumptions that the optical field remains continuous across the boundary and that, as is customary in physical optics, the effects of reflection can be neglected. The discontinuity may therefore be thought of merely as a limiting case of an optical element represented by an amplitude transmission function $t(\boldsymbol{\rho})$, with $t(\boldsymbol{\rho})$ approaching unity. Since the transmission function is independent of the properties of the incident field, such as its direction of propagation, this method typically leads to results that can be expected to hold only in the paraxial regime.

Our analysis showed that the radiance theorem with light of any state of coherence contains an additional factor, not present in the conventional radiance theorem, that depends in general both on the optical intensity and on the complex degree of spatial coherence of the light at the interface. For a quasi-homogeneous field, this factor becomes independent of the optical intensity distribution and is given, apart from a geometrical part $\cos\theta_2 \cos\theta_1$, by the [illegible] of the Fourier transform of the complex degree of spatial coherence evaluated at the [illegible] $\mathbf{f}_2 = k_2\mathbf{s}_2$ and $\mathbf{f}_1 = k_1\mathbf{s}_1$, respectively (see equation [illegible]) [illegible] clearly that the energy transmission across a medium disc[illegible] [illegible] on the state of coherence of the wavefield.

It was also shown that, for blackbody radiation fields, the [illegible] radiance theorem that we obtained is in relatively good [illegible] conventional radiance theorem over a wide range of angles, when [illegible] light may be assumed to be a reasonable approximation. [illegible] theorem with partially coherent light at larger angles of [illegible] use of the full electromagnetic theory, with proper [illegible] conditions at the discontinuity surface.

Acknowledgments

This work was carried out while the author held a visiting appoint[illegible] Department of Physics and Astronomy, University of Rochester [illegible]. The author wishes to thank Professor Emil Wolf for numerous helpful [illegible] concerning the subject matter of the present paper. The research was [illegible] the Air Force Geophysics Laboratory under AFOSR Task [illegible] and by the [illegible] Aaltonen Foundation.

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Invariance of the Spectrum of Light on Propagation

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(Received 27 January 1986)

The question is raised as to whether the normalized spectrum of light remains unchanged on propagation through free space. It is shown that for sources of a certain class that includes the usual thermal sources, the normalized spectrum will, in general, depend on the location of the observation point unless the degree of spectral coherence of the light across the source obeys a certain scaling law. *Possible implications of the analysis for astrophysics are mentioned*

PACS numbers: 42.10.Mg, 07.65.−b, 42.68.Hf

Measurements of the spectrum of light are generally made some distance away from its sources and in many cases, as for example in astronomy, they are made exceedingly far away. It is taken for granted that the normalized spectral distribution of the light incident on a detector after propagation from the source through free space is the same as that of the light in the source region. I will refer to this assumption as the assumption of *invariance of the spectrum on propagation*. This assumption, which is implicit in all of spectroscopy, does not appear to have been previously questioned, probably because with light from traditional sources one has never encountered any problems with it. However, with the gradual development of rather unconventional light sources and with the relatively frequent discoveries of stellar objects of an unfamiliar kind, it is obviously desirable to understand whether all such sources generate light whose spectrum is invariant on propagation, and if so, what the reasons for it are. Actually it is not difficult to conceive of sources that generate light whose spectrum is not invariant on propagation. In this note I will show what are the characteristics of a certain class of sources that generate light whose spectrum is invariant, at least in the far zone.

From the standpoint of optical coherence theory, invariance of the spectrum of light on propagation from conventional sources is a rather remarkable fact, as can be seen from the following simple argument. Consider an optical field generated by a stationary source in free space. The basic field variable, say the electric field strength at the space-time point $(\mathbf{r},t)$, may be represented by its complex analytic signal[1,2] $E(\mathbf{r},t)$. According to the Wiener-Khintchine theorem[3] the spectral density of the light at the point $\mathbf{r}$ is then represented by the Fourier transform,

$$S(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\Gamma(\mathbf{r},\tau)e^{i\omega\tau}\,d\tau, \tag{1}$$

of the autocorrelation function (known in the optical context as the self-coherence function) of the field variable. It is defined as

$$\Gamma(\mathbf{r},\tau)=\langle E^*(\mathbf{r},t)E(\mathbf{r},t+\tau)\rangle, \tag{2}$$

where the angular brackets denote the ensemble average. Now the spectral density and the self-coherence function are the "diagonal elements" $(\mathbf{r}_2=\mathbf{r}_1=\mathbf{r})$ of two basic optical correlation functions, viz., the cross-spectral density

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega)=\int_{-\infty}^{\infty}\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{i\omega\tau}\,d\tau, \tag{3}$$

and the mutual coherence function

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)=\langle E^*(\mathbf{r}_1,t)E(\mathbf{r}_2,t+\tau)\rangle. \tag{4}$$

It is well known that both the mutual coherence function and the cross-spectral density obey precise propagation laws. For example, in free space[4]

$$(\nabla_j^2+k^2)W(\mathbf{r}_1,\mathbf{r}_2,\omega)=0 \quad (j=1,2), \tag{5}$$

where

$$k=\omega/c, \tag{6}$$

with c being the speed of light *in vacuo* and ∇_j^2 being the Laplacian operator acting with respect to the variable $\mathbf{r}_j$. Consequently, both the mutual coherence function and the cross-spectral density and, in fact, also their normalized values change appreciably on propagation. For example, for a spatially incoherent planar source $W(\mathbf{r}_1,\mathbf{r}_2,\omega)$ and $\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)$ will be essentially δ correlated with respect to $\mathbf{r}_1$ and $\mathbf{r}_2$ at the source plane but will have nonzero values for widely separated pairs of points which are sufficiently far away from the source. This is the essence of the well known van Cittert–Zernike theorem (Ref. 1, Sect. 10.4.2). In physical terms, the correlation in the field generated by a spatially incoherent source may be shown to have its origin in the process of superposition. We thus have the following rather strange situation: The correlations of the light may change drastically on propagation; yet, under commonly occurring circumstances, their (suitably normalized) diagonal elements, which represent the spectrum of the light or its Fourier transform, remain unchanged.

To obtain some insight into this problem we consider light generated by a very simple model source, namely, a planar source occupying a finite domain D of a plane $z=0$ and radiating into the half space $z>0$, which has the same spectral distribution $S^{(0)}(\omega)$ at each source point $P(\boldsymbol{\rho})$ and whose degree of spectral coherence[5] $\mu^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)$ is statistically homogeneous, i.e., has the functional form $\mu^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega)$. The cross-spectral density of the light across the source plane is then given by

$$W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)$$
$$=\epsilon(\boldsymbol{\rho}_1)\epsilon(\boldsymbol{\rho}_2)S^{(0)}(\omega)\mu^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\omega), \quad (7)$$

where $\epsilon(\boldsymbol{\rho})=1$ or 0 according to whether the point $P(\boldsymbol{\rho})$ is located within or outside the source area D in the plane $z=0$.

We will also assume that at each effective frequency ω present in the source spectrum, the linear dimensions of the source are much larger than the spectral correlation length [the effective width Δ of $|\mu^{(0)}(\boldsymbol{\rho}',\omega)|$]. Sources of this kind belong to the class of so-called *quasihomogeneous sources*,[6] which have been extensively studied in coherence theory in recent years. Most of the usual thermal sources are of this kind.

The radiant intensity $J_\omega(\mathbf{u})$, i.e., the rate at which energy is radiated at frequency ω per unit solid angle around a direction specified by a unit vector $\mathbf{u}$, is given by the expression [cf. Ref. 6, Eq. (4.8)]

$$J_\omega(\mathbf{u})=k^2AS^{(0)}(\omega)\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega)\cos^2\theta \quad (8)$$

In this formula, A is the area of the source,

$$\tilde{\mu}^{(0)}(\mathbf{f},\omega)=\frac{1}{(2\pi)^2}\int\mu^{(0)}(\boldsymbol{\rho}',\omega)e^{-i\mathbf{f}\cdot\boldsymbol{\rho}'}d^2\rho' \quad (9)$$

is the two-dimensional spatial Fourier transform of the degree of spectral coherence, $\mathbf{u}_\perp$ is the transverse part of the unit vector $\mathbf{u}$, i.e., the component of $\mathbf{u}$ (considered as a two-dimensional vector) perpendicular to the z axis, and θ is the angle between the $\mathbf{u}$ and the z directions (see Fig. 1). Evidently the normalized spectral density $S^{(\infty)}(\mathbf{u},\omega)$ at a point in the far zone, in the direction specified by the unit vector $\mathbf{u}$, is given by

$$S^{(\infty)}(\mathbf{u},\omega)=J_\omega(u)/\int J_\omega(\mathbf{u})d\omega. \quad (10)$$

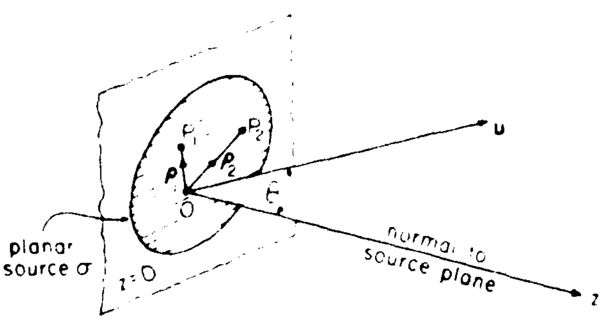

FIG. 1. Illustration of the notation.

On substituting Eq. (8) into Eq. (10) we obtain for the normalized spectrum in the far zone the expression

$$S^{(\infty)}(\mathbf{u},\omega)=\frac{k^2S^{(0)}(\omega)\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega)}{\int k^2S^{(0)}(\omega)\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega)d\omega}. \quad (11)$$

It is clear from Eq. (11) that the normalized spectrum of the light depends on the direction $\mathbf{u}$; i.e., it is in general not invariant throughout the far zone. However, it is seen at once from Eq. (11) that it will be invariant throughout the far zone if the Fourier transform of the degree of spectral coherence of the light in the source plane is the product of a function of frequency and a function of direction, i.e., it is of the form

$$\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega)=F(\omega)\tilde{H}(\mathbf{u}_\perp). \quad (12)$$

In this case Eq. (11) reduces to

$$S^{(\infty)}(\mathbf{u},\omega)=\frac{k^2S^{(0)}(\omega)F(\omega)}{\int k^2S^{(0)}(\omega)F(\omega)d\omega}, \quad (13)$$

and the expression on the right is independent of the direction $\mathbf{u}$.

I will now show that the condition (12) has some interesting implications, which follow from the fact that $\mu^{(0)}$ is a correlation coefficient. Before doing this we note that since $\mathbf{u}$ is a unit vector, $|\mathbf{u}_\perp|<1$. However, we will now assume that the factorization condition (12) holds for all two-dimensional vectors $\mathbf{u}_\perp$ ($0\leq|\mathbf{u}_\perp|<\infty$). This assumption will be trivially satisfied if the degree of spectral coherence $\mu^{(0)}(\boldsymbol{\rho}',\omega)$ is, at each effective temporal frequency ω, band limited in the spatial frequency plane to a circle of radius k about the origin; in more physical terms this condition means that $\mu^{(0)}(\boldsymbol{\rho}',\omega)$ does not vary appreciably over distances of the order of the wavelength $\lambda=2\pi/k$. With this being understood let us take the Fourier transform of Eq. (12). We then find at once that

$$\mu^{(0)}(\boldsymbol{\rho}',\omega)$$
$$=F(\omega)\int\tilde{H}(\mathbf{u}_\perp)\exp(ik\mathbf{u}_\perp\cdot\boldsymbol{\rho}')d^2(k\mathbf{u}_\perp), \quad (14)$$

i.e.,

$$\mu^{(0)}(\boldsymbol{\rho}',\omega)=k^2F(\omega)H(k\boldsymbol{\rho}'), \quad (15)$$

where H is, of course, the two-dimensional Fourier transform of $\tilde{H}$. Since $\mu^{(0)}(\boldsymbol{\rho}',\omega)$ is a correlation coefficient it has the value unity when $\boldsymbol{\rho}'=0$, i.e.,

$$\mu^{(0)}(0,\omega)=1, \text{ for all } \omega, \quad (16)$$

and hence Eq. (15) implies that

$$k^2F(\omega)=[H(0)]^{-1}. \quad (17)$$

Since the left-hand side of Eq. (17) depends on the frequency but the right-hand side is independent of it, each side must be a constant (α say) and consequently

$$F(\omega) = \alpha/k^2. \tag{18}$$

Two important conclusions follow at once from these results. If we substitute Eq. (18) into Eq. (13) we obtain the following expression for the normalized spectrum of light in the far zone

$$S^{(\infty)}(\mathbf{u},\omega) = S^{(\infty)}(\omega) = \frac{S^{(0)}(\omega)}{\int S^{(0)}(\omega)\,d\omega}. \tag{19}$$

This formula shows that not only is the normalized spectrum of the light now the same throughout the far zone, but it is also equal to the normalized spectrum of the light at each source point.

Next we substitute Eq. (18) into Eq. (15) and set $\alpha H = h$, $\rho = \rho_2 - \rho_1$. We then obtain for $\mu^{(0)}$ the expression

$$\mu^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \omega) = h[k(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)]$$

$$(k = \omega/c); \tag{20}$$

i.e., the complex degree of spectral coherence is a function of the variable $\xi = k(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)$ only. We will refer to Eq. (20) as the *scaling law*. Obviously for a source that satisfies this law, the knowledge of the degree of spectral coherence of the light in the source plane at any particular frequency ω specifies it for all frequencies.

The scaling law (20), which ensures that for sources of the class that we are considering the normalized spectrum of the light is the same throughout the far zone and is equal to the normalized spectrum of the light at each source point [Eq. (19)], is the main result of this note.

It is natural to inquire whether sources are known that obey this scaling law. The answer is affirmative. Many of the commonly occurring sources, including blackbody sources, obey Lambert's radiation law [Ref. 1, Sect. 4.8.1]. It is known[6] that all quasi-homogeneous Lambertian sources have the same degree of spectral coherence, viz.

$$\mu^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \omega) = \sin(k|\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1|)/k|\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1|, \tag{21}$$

which is seen to satisfy the scaling law (20). According to the preceding analysis such sources will generate light whose normalized spectrum is the same throughout the far zone and is equal to the normalized spectrum at each source point. This fact is undoubtedly largely responsible for the commonly held, but nevertheless incorrect, belief that spectral invariance is a general property of light.

This Letter has dealt with what is probably the simplest problem regarding spectral invariance on propagation. It would seem that some significant questions in this area might be profitably studied. Among them are the elucidation of the physical origin of the scaling law, spectral properties of light from a broader class of sources than considered here, the relation between the scaling law and Mandel's results regarding cross-spectrally pure light,[8,9] and relativistic effects. Applications of the results to problems of astrophysics might be of particular interest; at this stage one might only speculate whether source correlations may perhaps not give rise to differences between the spectrum of the emitted light and the spectrum of the detected light that originates in some stellar sources.

It is a pleasure to acknowledge stimulating discussions with Professor Leonard Mandel about the subject matter of this note. This research was supported by the National Science Foundation and by the Air Force Geophysics Laboratory under Air Force Office of Scientific Research Task No. 2310G1.

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[5]The degree of spectral coherence is defined by the formula [cf. L. Mandel and E. Wolf, J. Opt. Soc. Am. **66**, 529 (1976)]

$$\mu^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \frac{W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)}{[W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1, \omega)]^{1/2}[W^{(0)}(\boldsymbol{\rho}_2, \boldsymbol{\rho}_2, \omega)]^{1/2}}$$

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RADIOMETRY AS A SHORT-WAVELENGTH LIMIT OF STATISTICAL WAVE THEORY WITH GLOBALLY INCOHERENT SOURCES

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Received 22 May 1985

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1. Introduction

During the last two decades several attempts have been made to elucidate the foundations of radiometry. In particular several authors [1–5] proposed expressions for the basic quantity of radiometry, namely the (spectral) radiance, in terms of various second-order correlation functions of the optical field. Although each of the proposed expressions exhibits some of the well-known properties that are attributed to the radiance in traditional radiometry, none of them possesses all of them, for sources and fields of arbitrary state of coherence. In particular, some of the proposed expressions for the radiance can take on negative values, a result that contradicts the physical meaning of radiance. More recently it was shown [6] that it is not possible to define a radiance for a planar source which depends linearly on a second-order correlation function of the source field and which satisfies three basic postulates of radiometry for every possible state of coherence of the source [‡1].

We believe that the difficulties just mentioned arose because the previous investigations did not take into account the fact that traditional radiometry deals with sources that are spatially highly incoherent (namely thermal sources) *and* that they generate radiation whose effective wavelengths λ are very small compared with their linear dimensions [‡2]. We show in this note that when these facts are taken into account a consistent formulation of radiometry is obtained on the basis of second-order coherence theory, at least for sources and fields in free space. More specifically, we show that traditional radiometry correctly describes the behavior of fields generated by planar quasi-homogeneous sources [11] in free space, in the asymptotic limit as the wave number $k = 2\pi/\lambda \to \infty$.

Research supported by the National Science Foundation under Grant #PHY-8314626 and the Air Force Geophysics Laboratory under AFOSR Task 2310G1.

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[‡1] In an interesting recent paper [7] a definition of radiance was proposed which depends non-linearly on a second-order correlation function of the source and which satisfies the three postulates. It appears, however, that this radiance does not obey the radiometric law for the propagation of radiance in free space.

[‡2] Allusions to the possibility that traditional radiometry implies such restrictions have been made from time to time [4b], [8–10], but the appropriate mathematical justification has not been previously provided.

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2. Generalized radiance

Let us consider a secondary source σ, occupying a finite portion of the plane $z = 0$ and radiating into the half-space $z > 0$. We assume that the source fluctuations are statistically stationary. We will denote by $\boldsymbol{\rho}$ the two-dimensional vector specifying the location of a source point S and by $\boldsymbol{r}$ the three-dimensional vector specifying the location of a field point P in the half-space $z > 0$, both referred to a fixed origin O in the source region (see fig. 1).

Let $W(\boldsymbol{r}_1, \boldsymbol{r}_2, \nu)$ be the cross-spectral density of the field generated by the source at two points P_1 and P_2. It is known that the cross-spectral density may be represented in terms of an ensemble of monochromatic wave fields $\{U(\boldsymbol{r}, \nu)\exp(-2\pi i\nu t)\}$, all of the same frequency ν, as [12]

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2, \nu) = \langle U^*(\boldsymbol{r}_1, \nu)\, U(\boldsymbol{r}_2, \nu)\rangle, \tag{2.1}$$

where the angular bracket on the right-hand side of eq. (2.1) denotes the average over this ensemble. The space dependent part of each member of this ensemble obeys, throughout the half-space $z > 0$, the Helmholtz equation

$$(\nabla^2 + k^2)\, U(\boldsymbol{r}, \nu) = 0, \tag{2.2}$$

where $k = 2\pi\nu/c$ and c is the speed of light in vacuo, and it behaves as an outgoing spherical wave at infinity in this half-space. As is well known, such solutions can be represented, under very general conditions, in the form of an angular spectrum of plane waves, i.e. in the form [13]

$$U(\boldsymbol{r}, \nu) = \int a(\boldsymbol{s}_\perp, \nu)\exp(ik\boldsymbol{s}\cdot\boldsymbol{r})\, \mathrm{d}^2 s_\perp, \tag{2.3}$$

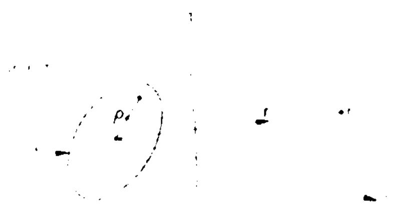

Fig. 1. Illustrating the notation.

where

$$\boldsymbol{s} \equiv (s_x, s_y, s_z), \qquad \boldsymbol{s}_\perp \equiv (s_x, s_y, 0), \tag{2.4}$$

$$\begin{aligned} s_z &= +(1 - s_\perp^2)^{1/2} && \text{if } |\boldsymbol{s}_\perp| \le 1, \\ &= +i(s_\perp^2 - 1)^{1/2} && \text{if } |\boldsymbol{s}_\perp| > 1, \end{aligned} \tag{2.5}$$

and the integration on the right-hand side of eq. (2.3) extends over the whole $s_\perp$-plane $(0 \le s_x^2 + s_y^2 < \infty)$.

In the above notation the complex version of the generalized radiance function introduced by Walther in refs. [4] may be written in the form

$$B_\nu(\boldsymbol{r}, \boldsymbol{s}) = s_z\langle U^*(\boldsymbol{r}, \nu)\, a(\boldsymbol{s}_\perp, \nu)\rangle\exp(ik\boldsymbol{s}\cdot\boldsymbol{r}) \tag{2.6}$$

For purposes of later discussion we will express $B_\nu(\boldsymbol{r}, \boldsymbol{s})$ in terms of the values $B_\nu^{(0)}(\boldsymbol{\rho}, \boldsymbol{s})$, which it takes on the source plane $z = 0$. This can readily be done by making use of the fact that the outgoing solution of the Helmholtz equation may be expressed in terms of its boundary value $U^{(0)}(\boldsymbol{\rho}, \nu)$ in the plane $z = 0$ by the Rayleigh formula [14]

$$U(\boldsymbol{r}, \nu) = \int_\sigma G(\boldsymbol{R}, \nu)\, U^{(0)}(\boldsymbol{\rho}, \nu)\, \mathrm{d}^2\rho, \tag{2.7}$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{\rho}$ and $G(\boldsymbol{R}, \nu)$ is the Green's function

$$G(\boldsymbol{R}, \nu) = -(1/2\pi)(\partial/\partial z)[\exp(ikR)/R], \tag{2.8}$$

with $R = |\boldsymbol{R}|$. On substituting for $U(\boldsymbol{r}, \nu)$ from eq. (2.7) into eq. (2.6) and interchanging the orders of integration and averaging one readily finds that

$$B_\nu(\boldsymbol{r}, \boldsymbol{s}) = \exp(ik\boldsymbol{s}\cdot\boldsymbol{r})\int_\sigma G^*(\boldsymbol{R}, \nu) \times B_\nu^{(0)}(\boldsymbol{\rho}, \boldsymbol{s})\exp(-ik\boldsymbol{s}_\perp\cdot\boldsymbol{\rho})\, \mathrm{d}^2\rho, \tag{2.9}$$

where $B_\nu^{(0)}(\boldsymbol{\rho}, \boldsymbol{s})$ is the generalized radiance at a point in the source plane, viz.,

$$B_\nu^{(0)}(\boldsymbol{\rho}, \boldsymbol{s}) = s_z\langle U^{(0)*}(\boldsymbol{\rho}, \nu)\, a(\boldsymbol{s}_\perp, \nu)\rangle\exp(ik\boldsymbol{s}_\perp\cdot\boldsymbol{\rho}). \tag{2.10}$$

In general $B_\nu(\boldsymbol{r}, \boldsymbol{s})$ is complex. Hence it cannot have the physical significance of radiance. The same is true of the real part of $B_\nu(\boldsymbol{r}, \boldsymbol{s})$, because, as was shown elsewhere [15], it can sometimes take on negative values. However, as shown in ref. [4a] its real part yields the correct value for the radiant intensity via one of the standard formulas of traditional radiometry.

3. Generalized radiance of a field generated by a quasi-homogeneous source

We will now specialize the expressions (2.10) and (2.9) for the generalized radiance to the case when the source is quasi-homogeneous [11]. The degree of spectral coherence of a quasi-homogeneous source depends on the two source variables $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2$ only through the difference $\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$. Consequently its cross-spectral density has the form

$$W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu) = [I^{(0)}(\boldsymbol{\rho}_1, \nu)]^{1/2}[I^{(0)}(\boldsymbol{\rho}_2, \nu)]^{1/2} g^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \nu), \tag{3.1}$$

where $I^{(0)}(\boldsymbol{\rho}, \nu)$ represents the optical intensity at the source point S and $g^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \nu)$ is the degree of spectral coherence of the light at two source points S_1 and S_2 (with position vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ respectively). Moreover, for sources of this class $I^{(0)}(\boldsymbol{\rho}, \nu)$ changes so slowly with the position ($\boldsymbol{\rho}$) across the source that it is essentially constant over regions whose linear dimensions are of the order of the effective range of $g^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \nu)$, i.e. of the order of the spectral correlation length, l_ν say, of the light across the source. It is also assumed that the linear dimensions of the source are large compared both with l_ν and with the wavelength $\lambda = c/\nu$.

Sources of this class include the usual thermal (e.g. blackbody) sources for which l_ν is of the order of the wavelength, but other types of sources, for which l_ν may be much greater than the wavelength, also belong to this category. However, all quasi-homogeneous sources may be said to be *globally incoherent*, since the domain which they occupy is very much larger than their (spectral) coherence area ($\approx \pi l_\nu^2$).

To determine the generalized radiance of the field produced by a quasi-homogeneous source we proceed as follows. We first set $z = 0$ in eq. (2.3) and then take the Fourier inverse of the resulting formula. This gives an expression for $a(\boldsymbol{s}_\perp, \nu)$ in terms of the boundary values, $U^{(0)}(\boldsymbol{\rho}, \nu)$, of $U(\boldsymbol{r}, \nu)$ in the plane $z = 0$. Next we substitute this expression into eq. (2.10) and obtain the following expression for the generalized radiance in the source plane:

$$\mathcal{B}_\nu^{(0)}(\boldsymbol{\rho}, \boldsymbol{s}) = (k/2\pi)^2 s_z \exp(\mathrm{i}k\boldsymbol{s}_\perp \cdot \boldsymbol{\rho}) \times \int_\sigma W^{(0)}(\boldsymbol{\rho}, \boldsymbol{\rho}', \nu) \exp(-\mathrm{i}k\boldsymbol{s}_\perp \cdot \boldsymbol{\rho}')\, \mathrm{d}^2\rho', \tag{3.2}$$

where $W^{(0)}(\boldsymbol{\rho}, \boldsymbol{\rho}', \nu)$ is the cross-spectral density of the field in the source plane. In deriving eq. (3.2), eq. (2.1) was used.

For a quasi-homogeneous source $W^{(0)}$ is given by eq. (3.1), and if we use that equation the formula (3.2) becomes

$$\mathcal{B}_\nu^{(0)}(\boldsymbol{\rho}, \boldsymbol{s}) = (k/2\pi)^2 s_z \exp(\mathrm{i}k\boldsymbol{s}_\perp \cdot \boldsymbol{\rho}) \times [I^{(0)}(\boldsymbol{\rho}, \nu)]^{1/2} \int_\sigma [I^{(0)}(\boldsymbol{\rho}', \nu)]^{1/2} g^{(0)}(\boldsymbol{\rho}' - \boldsymbol{\rho}, \nu) \times \exp(-\mathrm{i}k\boldsymbol{s}_\perp \cdot \boldsymbol{\rho}')\, \mathrm{d}^2\rho'. \tag{3.3}$$

Since for a quasi-homogeneous source the optical intensity $I^{(0)}(\boldsymbol{\rho}, \nu)$ (with ν fixed) remains sensibly constant over regions whose linear dimensions are of the order of the effective range l_ν of $g^{(0)}$, we may replace the factor $[I^{(0)}(\boldsymbol{\rho}', \nu)]^{1/2}$ by $[I^{(0)}(\boldsymbol{\rho}, \nu)]^{1/2}$ in eq. (3.3) and then take it outside the integral sign. Moreover, since the linear dimensions of a quasi-homogeneous source are much greater than l_ν, the integration over σ may be taken over the whole $\boldsymbol{\rho}'$-plane without introducing an appreciable error. Eq. (3.3) then gives, with very high degree of accuracy, the following expression for $\mathcal{B}_\nu^{(0)}(\boldsymbol{\rho}, \boldsymbol{s})$:

$$\mathcal{B}_\nu^{(0)}(\boldsymbol{\rho}, \boldsymbol{s}) = k^2 s_z I^{(0)}(\boldsymbol{\rho}, \nu) \tilde{g}^{(0)}(k\boldsymbol{s}_\perp, \nu). \tag{3.4}$$

Here $\tilde{g}^{(0)}(\boldsymbol{f}, \nu)$ is the two-dimensional Fourier transform of $g^{(0)}(\boldsymbol{\rho}', \nu)$, i.e.

$$\tilde{g}^{(0)}(\boldsymbol{f}, \nu) = (2\pi)^{-2} \int g^{(0)}(\boldsymbol{\rho}', \nu) \exp(-\mathrm{i}\boldsymbol{f} \cdot \boldsymbol{\rho}')\, \mathrm{d}^2\rho'. \tag{3.5}$$

The formula (3.4) shows that the behavior of the generalized radiance of a quasi-homogeneous source at a point S in the source plane, in a direction specified by the unit vector $\boldsymbol{s}$, is determined by the value of the optical intensity at that point and by the spatial Fourier component labeled by the spatial-frequency vector $k\boldsymbol{s}_\perp$ of the degree of spectral coherence of the light in the source plane. This result was obtained previously by a slightly different argument in ref. [11], eq. (A10).

An expression for the generalized radiance of the field generated by the quasi-homogeneous source at any point P the half-space $z > 0$ is obtained at once on substituting from eq. (3.4) into eq. (2.9) and one finds that

$$B_\nu(\boldsymbol{r}, \boldsymbol{s}) = k^2 s_z \tilde{g}^{(0)}(k\boldsymbol{s}_\perp, \nu) C_\nu^*(\boldsymbol{r}, \boldsymbol{s}_\perp) \exp(ik\boldsymbol{s}\cdot\boldsymbol{r}), \quad (3.6)$$

where

$$C_\nu(\boldsymbol{r}, \boldsymbol{s}_\perp) = \int_\sigma G(\boldsymbol{R}, \nu) I^{(0)}(\boldsymbol{\rho}, \nu) \exp(ik\boldsymbol{s}_\perp\cdot\boldsymbol{\rho})\, d^2\rho. \quad (3.7)$$

4. The asymptotic limit $k \to \infty$ of the generalized radiance for a field generated by a quasi-homogeneous source

Let us now consider the behavior of the expression (3.6) for very short wavelengths λ or, more precisely, determine its asymptotic limit as the wave number $k = 2\pi/\lambda \to \infty$. For this purpose we carry out the differentiation on the right-hand side of eq. (2.8) and substitute the resulting expression for the Green's function $G(\boldsymbol{R}, \nu)$ in eq. (3.7). We then find that

$$C_\nu(\boldsymbol{r}, \boldsymbol{s}_\perp) = C_\nu^{(1)}(\boldsymbol{r}, \boldsymbol{s}_\perp) + C_\nu^{(2)}(\boldsymbol{r}, \boldsymbol{s}_\perp), \quad (4.1)$$

where

$$C_\nu^{(1)}(\boldsymbol{r}, \boldsymbol{s}_\perp) = \frac{kz}{2\pi i}\int_\sigma I^{(0)}(\boldsymbol{\rho}, \nu)\frac{\exp[ik\phi(\boldsymbol{R}, \boldsymbol{\rho})]}{R^2}\, d^2\rho, \quad (4.2)$$

$$C_\nu^{(2)}(\boldsymbol{r}, \boldsymbol{s}_\perp) = \frac{z}{2\pi}\int_\sigma I^{(0)}(\boldsymbol{\rho}, \nu)\frac{\exp[ik\phi(\boldsymbol{R}, \boldsymbol{\rho})]}{R^3}\, d^2\rho, \quad (4.3)$$

and

$$\phi(\boldsymbol{R}, \boldsymbol{\rho}) = R + \boldsymbol{s}_\perp\cdot\boldsymbol{\rho}. \quad (4.4)$$

Each of the integrals in eqs. (4.2) and (4.3) depends on k in two ways: via the exponential term $\exp[ik\phi(\boldsymbol{R}, \boldsymbol{\rho})]$ and via the k-dependence implicit in the optical intensity $I^{(0)}(\boldsymbol{\rho}, \nu) = I^{(0)}(\boldsymbol{\rho}, kc/2\pi)$. As k becomes larger and larger, the exponential term will, in general, oscillate more and more rapidly as the point S explores the domain of integration. On the other hand, for any fixed value of k, the optical intensity of a quasi-homogeneous source varies slowly with $\boldsymbol{\rho}$; hence its k-dependence may be neglected in the asymptotic evaluation of $C_\nu^{(1)}$ and $C_\nu^{(2)}$, as is clear from the principle of stationary phase [16]. Moreover, it is evident from comparison of the expressions on the right-hand sides of eqs. (4.2) and (4.3) that as $k \to \infty$, $C_\nu^{(2)}$ is of higher order in $1/k$ than $C_\nu^{(1)}$. Hence we only need to confine our attention to the asymptotic approximation to $C_\nu^{(1)}$.

Straightforward application of the principle of stationary phase shows that, in general, the integral in eq. (4.2) has either one critical point of the first kind or none at all. Let us set

$$\boldsymbol{r} = (x, y, z), \qquad \boldsymbol{r}_\perp = (x, y, 0) \quad (4.5)$$

and let us denote by S_0 the point specified by the position vector

$$\boldsymbol{\rho}_0 = \boldsymbol{r}_\perp - (z/s_z)\boldsymbol{s}_\perp, \quad (4.6)$$

which may readily be shown to lie in the plane $z = 0$. One finds that if S_0 lies within the source domain σ, it is the critical point of the first kind, and that if S_0 lies outside σ, the integral does not have a critical point of the first kind. We will see shortly that the S_0 has a simple geometrical significance.

When S_0 is located within σ, the asymptotic approximation to $C_\nu(\boldsymbol{r}, \boldsymbol{s})$ is found to be

$$C_\nu(\boldsymbol{r}, \boldsymbol{s}_\perp) \sim I^{(0)}(\boldsymbol{r}_\perp - (z/s_z)\boldsymbol{s}_\perp, \nu)\exp(ik\boldsymbol{s}\cdot\boldsymbol{r}) \quad \text{as } k \to \infty. \quad (4.7a)$$

When S_0 is located outside σ, the asymptotic approximation comes from contributions of critical points of the second kind, and is of higher order in $1/k$ than the expression on the right-hand side of eq. (4.7a) and we may express this fact (taking some liberty with the interpretation of the asymptotic symbol) by writing

$$C_\nu(\boldsymbol{r}, \boldsymbol{s}_\perp) \sim 0 \quad \text{as } k \to \infty. \quad (4.7b)$$

On substituting from eqs. (4.7) into eq. (3.6) and using the fact that $I^{(0)}$ is zero when its argument lies outside of σ, we finally obtain the following asymptotic approximation to the generalized radiance function of a field generated by a quasi-homogeneous source:

$$B_\nu(\boldsymbol{r}, \boldsymbol{s}) \sim \mathcal{B}_\nu(\boldsymbol{r}, \boldsymbol{s}) \quad \text{as } k \to \infty, \quad (4.8)$$

where

$$B_\nu(\boldsymbol{r},\boldsymbol{s}) \approx k^2 s_z I^{(0)}(\boldsymbol{r}_\perp - (z/s_z)\boldsymbol{s}_\perp, \nu)\, \tilde{g}^{(0)}(k\boldsymbol{s}_\perp, \nu). \quad (4.9)$$

The asymptotic approximation (4.8)–(4.9) to the generalized radiance is the main result of this note. We will show that it has a number of important consequences.

First we note that according to traditional radiometry the rate at which energy crosses an area element dA per unit solid angle around a direction specified by a real unit vector $\boldsymbol{s}$ is given by $B_\nu(\boldsymbol{r},\boldsymbol{s})\boldsymbol{s}\cdot\boldsymbol{n}\, dA$, where $\boldsymbol{n}$ is the unit normal to dA. In particular it follows from this formula that the rate at which energy is radiated into the far zone per unit solid angle around the $\boldsymbol{s}$-direction (i.e. the radiant intensity)[‡3] across any plane $z = z_0 = \text{const.} > 0$ is given by

$$\mathcal{J}_\nu(\boldsymbol{s}) = s_z \int_{z=z_0} B_\nu(\boldsymbol{r},\boldsymbol{s})\, dx\, dy. \quad (4.10)$$

On substituting from eq. (4.9) into eq. (4.10) we readily find that

$$\mathcal{J}_\nu(\boldsymbol{s}) = (2\pi k)^2 s_z^2\, \tilde{I}^{(0)}(0,\nu)\tilde{g}^{(0)}(k\boldsymbol{s}_\perp, \nu), \quad (4.11)$$

where

$$\tilde{I}^{(0)}(0,\nu) = (2\pi)^{-2} \int I^{(0)}(\boldsymbol{\rho},\nu)\, d^2\rho. \quad (4.12)$$

If we recall that $s_z = \cos\theta$, where θ is the angle that the (real) $\boldsymbol{s}$-direction makes with the normal to the source plane, the right-hand side of the formula (4.11) is found to be precisely the expression for radiant intensity from a quasi-homogeneous source, calculated by physical optics [ref. [11], eq. (4.8)].

It will be convenient for the purpose of subsequent discussion to express the formula (4.9) in two alternative forms. First we rewrite it as

$$B_\nu(\mathrm{P},\boldsymbol{s}) \approx k^2 s_z I^{(0)}(\mathrm{S}_0,\nu)\tilde{g}^{(0)}(k\boldsymbol{s}_\perp,\nu) \quad \text{if } \mathrm{S}_0 \in \sigma$$
$$= 0 \quad \text{if } \mathrm{S}_0 \notin \sigma \quad (4.13)$$

Further it follows from elementary geometry that the

[‡3] Rigorous justification for the identification of the expression (4.10) with the radiant intensity of physical optics requires some additional considerations, which we plan to present in another paper.

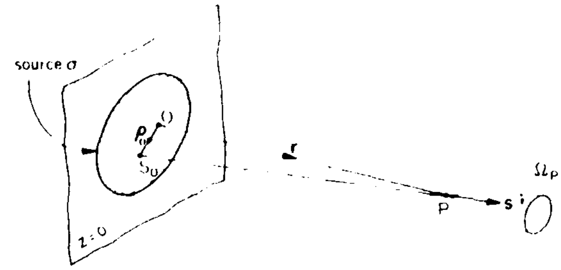

Fig. 2. Illustrating the notation relating to the formulas (4.14). S_0 is the point in the source plane whose position vector $\boldsymbol{\rho}_0$ is given by eq. (4.6); it is the point of intersection with the source plane of the line through the point P in the direction of the real unit vector $\boldsymbol{s}$.

point S_0, whose position vector is given by eq. (4.6), is precisely the point at which the line through P, in the direction specified by the unit vector $\boldsymbol{s}$ (again assumed to be real), intersects the source plane $z = 0$. Hence eq. (4.13) implies that

$$B_\nu(\mathrm{P},\boldsymbol{s}) = k^2 s_z I^{(0)}(\mathrm{S}_0,\nu)\tilde{g}^{(0)}(k\boldsymbol{s}_\perp,\nu) \quad \text{if } \boldsymbol{s} \in \Omega_\mathrm{P}$$
$$= 0 \quad \text{if } \boldsymbol{s} \notin \Omega_\mathrm{P} \quad (4.14)$$

where Ω_P denotes the solid angle generated by the lines pointing from the source points to P (see fig. 2).

The first three terms on the right-hand side of the first line of eq. (4.14) are evidently non-negative. So is the last term $\tilde{g}^{(0)}(k\boldsymbol{s}_\perp,\nu)$, since it is the Fourier transform of a non-negative definite function [17]. Hence

$$B_\nu(\mathrm{P},\boldsymbol{s}) \geq 0. \quad (4.15)$$

Let $B_\nu^{(0)}$ denote the limiting value of B_ν when the spatial argument ($\boldsymbol{r}$ or P) approaches the source plane $z = 0$. Since the optical intensity is zero at any point P *in that plane* which is located outside the source area σ, we have from eq. (4.13)

$$B_\nu^{(0)}(\mathrm{P},\boldsymbol{s}) = 0 \quad \text{if } \mathrm{P} \notin \sigma. \quad (4.16)$$

Finally we see at once from eq. (4.14) that

$$B_\nu(\mathrm{P},\boldsymbol{s}) = B_\nu^{(0)}(\mathrm{S}_0,\boldsymbol{s}) \quad (4.17)$$

This formula implies that $B_\nu(\mathrm{P},\boldsymbol{s})$ is constant along each line in the half-space $z > 0$.

The fact that eq. (4.10), with B_ν given by eq. (4.9), represents the radiant intensity as calculated from physical optics, as well as the results expressed by eqs. (4.15), (4.16) and (4.17), show that B_ν has all the main properties attributed to radiance. We may, therefore, conclude that *traditional radiometry, with the radiance given by eq. (4.9), correctly describes the behavior of fields generated by quasi-homogeneous planar sources in free space, in the asymptotic limit as* $k \equiv 2\pi/\lambda \to \infty$.

Finally we wish to remark that although we derived the expression (4.9) for the radiance from one particular definition of a generalized radiance function (introduced in refs. [4]), we believe that the same expression will follow, in the asymptotic limit of large wave number, from some of the other (non-equivalent) definitions of generalized radiance functions, when they are specialized to fields generated by quasi-homogeneous sources.

Acknowledgement

We wish to acknowledge stimulating discussions with Prof. W.T. Welford, F.R.S. and with Prof. R. Winston about the subject matter of this note.

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Radiance functions that depend nonlinearly on the cross-spectral density

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a reprint from **Journal of the Optical Society of America A**
volume 2, number 9, September 1985

Radiance functions that depend nonlinearly on the cross-spectral density

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Received March 20, 1985; accepted May 6, 1985

Recently a new definition of radiance was proposed [J. Opt. Soc. Am. A 1, 556 (1984)] that depends nonlinearly on the cross-spectral density of the field and satisfies the three major postulates of traditional radiometry. We show that there are an infinite number of such radiance functions. Their utility is discussed.

It is well known that one of the main problems encountered in the attempt to connect the theory of partial coherence with traditional radiometry[1-3] is that there is no radiance function that depends linearly on the cross-spectral density of the field and satisfies the three major postulates of traditional radiometry for planar sources of any state of coherence.[4,5] More specifically, consider a planar source of finite area D located in the plane $z = 0$ that emits light into the half-space $z > 0$. Let $\mathbf{r}$ be a two-dimensional position vector in the plane $z = 0$, $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ be the cross-spectral density in that plane, and $\mathbf{s}$ be a three-dimensional unit vector whose z component is nonnegative. Friberg[5] showed that there is no radiance function $B(\mathbf{r}, \mathbf{s}, \nu)$ that satisfies the following four conditions for planar sources of any state of coherence:

(I) $B(\mathbf{r}, \mathbf{s}, \nu)$ depends linearly on $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$,

(II) $B(\mathbf{r}, \mathbf{s}, \nu) \geq 0$ for all $\mathbf{r}$ and $\mathbf{s}$,

(III) $B(\mathbf{r}, \mathbf{s}, \nu) = 0$ when $\mathbf{r} \notin D$,

(IV) $\cos\theta \int_D B(\mathbf{r}, \mathbf{s}, \nu)\mathrm{d}^2 r = J(\mathbf{s}, \nu)$,

where $\cos\theta = \mathbf{s} \cdot \hat{z}$ and $J(\mathbf{s}, \nu)$ is the radiant intensity of physical optics. For example, Walther's two definitions[1,2] fail to satisfy requirement (II) for certain types of sources.[3,6]

In an interesting recent paper[7] a new definition of radiance was introduced that (a) depends nonlinearly on $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ and (b) satisfies requirements (II)–(IV) for sources of any state of coherence. *The purpose of this Communication is to show* that the radiance function of Ref. 7 is not unique in these respects. By using the methods of Ref. 7 we will show that there are an infinite number of radiance functions that satisfy conditions (a) and (b) above.

The cross-spectral density of the source can be represented by the Mercer expansion[8]

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \sum_n \lambda_n(\nu)\phi_n{}^*(\mathbf{r}_1, \nu)\phi_n(\mathbf{r}_2, \nu), \tag{1}$$

where the $\phi_n(\mathbf{r}, \nu)$ and $\lambda_n(\nu)$ are, respectively, the eigenfunctions and the eigenvalues of the Fredholm integral equation

$$\int_D W(\mathbf{r}_1, \mathbf{r}_2, \nu)\phi_n(\mathbf{r}_1, \nu)\mathrm{d}^2\mathbf{r}_1 = \lambda_n(\nu)\phi_n(\mathbf{r}_2, \nu). \tag{2}$$

The eigenfunctions are orthonormal over the domain D, i.e.,

$$\int_D \phi_n{}^*(\mathbf{r}, \nu)\phi_m(\mathbf{r}, \nu)\mathrm{d}^2\mathbf{r} = \delta_{nm}, \tag{3}$$

and the eigenvalues are real and nonnegative. Expansion (1) holds irrespective of whether the set of functions $\{\phi_n(r, \nu)\}$ is complete in the Hilbert space of functions that are square integrable over D.[8,9]

In Ref. 7 the following definition of radiance was proposed:

$$B(\mathbf{r}, \mathbf{s}, \nu) = \left(\frac{k}{2\pi}\right)^2 \cos\theta \left| \int_D \exp(ik\mathbf{s}\cdot\mathbf{r}')G(\mathbf{r}', \mathbf{r}, \nu)\mathrm{d}^2\mathbf{r}' \right|^2, \tag{4}$$

where $G(\mathbf{r}', \mathbf{r}, \nu)$, the generating function, was given by

$$G(\mathbf{r}', \mathbf{r}, \nu) = \chi_D(\mathbf{r}) \sum_n \sqrt{\lambda_n(\nu)}\phi_n(\mathbf{r}', \nu)\phi_n{}^*(\mathbf{r}, \nu) \tag{5}$$

and

$$\chi_D(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in D \\ 0, & \mathbf{r} \notin D \end{cases}. \tag{6}$$

Equation (4) forces $B(\mathbf{r}, \mathbf{s}, \nu)$ to be nonnegative; therefore condition (II) is fulfilled. Since

$$G(\mathbf{r}', \mathbf{r}, \nu) = 0, \qquad \mathbf{r} \notin D, \tag{7}$$

condition (III) is fulfilled. By using Eqs. (1), (3), and (5) it is a straightforward matter to show that

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \int_D G^*(\mathbf{r}_1, \mathbf{r}, \nu)G(\mathbf{r}_2, \mathbf{r}, \nu)\mathrm{d}^2\mathbf{r}, \tag{8}$$

and it follows from Eqs. (4) and (8) that condition (IV) is fulfilled (see Ref. 7 for details).

We will now show that for a given cross-spectral density function there are an infinite number of generating functions $G(\mathbf{r}', \mathbf{r}, \nu)$ that obey Eqs. (7) and (8). The corresponding ra

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diance functions obtained by using Eq. (4) will therefore depend nonlinearly on $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$, obey conditions (II)–(IV) and, in general, be different from the radiance function of Ref. 7.

Consider the expansion

$$G(\mathbf{r}', \mathbf{r}, \nu) = \chi_D(\mathbf{r}) \sum_m \sum_n a_{mn}(\nu)\phi_n(\mathbf{r}', \nu)\phi_m{}^*(\mathbf{r}, \nu), \quad (9)$$

where[10]

$$\sum_m \sum_n |a_{mn}(\nu)|^2 < \infty. \quad (10)$$

$G(\mathbf{r}', \mathbf{r}, \nu)$ satisfies Eq. (7). By using Eqs. (9) and (3) one obtains

$$\int_D G^*(\mathbf{r}_1, \mathbf{r}, \nu)G(\mathbf{r}_2, \mathbf{r}, \nu)\mathrm{d}^2\mathbf{r}$$

$$= \sum_m \sum_n \sum_{n'} a_{mn}{}^*(\nu)a_{mn'}(\nu)\phi_n{}^*(\mathbf{r}_1, \nu)\phi_{n'}(\mathbf{r}_2, \nu). \quad (11)$$

Equation (1) can be rewritten as

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \sum_n \sum_{n'} \lambda_n(\nu)\delta_{nn'}\phi_n{}^*(\mathbf{r}_1, \nu)\phi_{n'}(\mathbf{r}_2, \nu). \quad (12)$$

It follows from Eqs. (11), (12), and (3) that the generating function defined by Eq. (9) obeys Eq. (8) if and only if the expansion coefficients satisfy the scaled unitarity condition

$$\sum_m a_{mn}{}^*(\nu)a_{mn'}(\nu) = \lambda_n(\nu)\delta_{nn'} \quad (13)$$

Therefore any radiance function of the form (1), where the generating function is of the form (9) and the coefficients obey inequality (10) and Eq. (13), depends nonlinearly on the cross-spectral density in the source plane and satisfies conditions (II)–(IV). The radiance function of Ref. 7 corresponds to the choice

$$a_{mn}(\nu) = \sqrt{\lambda_m(\nu)}\delta_{mn} \quad (14)$$

Another simple choice would be

$$a_{mn}(\nu) = \sqrt{\lambda_m(\nu)}\exp[i\alpha_m(\nu)]\delta_{mn}, \quad (15)$$

where each $\alpha_m(\nu)$ is real.

The above result brings to mind two questions. First, of all the possible nonlinear radiance functions that are possible, is there one (e.g., the Hermitian one of Ref. 7) that is preferable? Unless one imposes additional physical restrictions on the problem, the answer to this question is clearly no.

Second, are these nonlinear radiance functions preferable to the two definitions of Walther? This is an open question at this time; however, the following points are relevant. Each of Walther's radiance functions is nonnegative when the source is quasi homogeneous.[11] Since quasi-homogeneous sources are globally spatially incoherent, Walther's radiance functions behave properly for the types of sources (incoherent) for which traditional radiometry was developed. Also, for certain types of sources Walther's radiance functions obey, approximately, the equation of radiative transfer when they propagate into the half-space $z > 0$.[1,2,12] Since $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ propagates into the half-space $z > 0$ according to two linear partial differential equations (the Helmholtz equations on the variables $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively), the nonlinear radiance functions may not propagate in this simple manner. Nevertheless, the nonlinear radiance functions are interesting and deserve further study.

ACKNOWLEDGMENTS

The authors would like to acknowledge their appreciation to E. Wolf for several stimulating discussions and helpful comments during the course of this work.

This research was supported by the U.S. Air Force Geophysics Laboratory under AFOSR Task 2310G1.

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